

CONVERGENCE PROPERTIES OF PSEUDO-MARGINAL MARKOV CHAIN MONTE CARLO ALGORITHMS

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ABSTRACT. We study convergence properties of pseudo-marginal Markov chain Monte Carlo algorithms [Andrieu and Roberts, *Ann. Statist.* **37** (2009) 697–725]. We find that the asymptotic variance of the pseudo-marginal algorithm is always at least as large as that of the marginal algorithm. We show that if the marginal chain is geometrically ergodic and the weights (normalised estimates of the target density) are uniformly bounded, then the pseudo-marginal chain is geometric. We consider also unbounded weight distributions and recover polynomial convergence rates in more specific cases, when the marginal algorithm is uniformly ergodic, an independent Metropolis-Hastings or a random-walk Metropolis targeting a super-exponential density with regular contours. Our results on geometric and polynomial convergence rates imply central limit theorems. We also prove that under general conditions, the asymptotic variance of the pseudo-marginal algorithm converges to the asymptotic variance of the marginal algorithm if the accuracy of the estimators is increased.

1. INTRODUCTION

Assume that one is interested in sampling from a probability distribution π defined on some measurable space $(X, \mathcal{B}(X))$. One practical recipe to achieve this in complex scenarios consists of using Markov chain Monte Carlo (MCMC) methods, of which the Metropolis-Hastings update is the main workhorse [12, 19]. We may write the Markov kernel related to a Metropolis-Hastings algorithm in the form

$$(1) \quad P(x, dy) := \min\{1, r(x, y)\} q(x, dy) + \delta_x(dy) \rho(x),$$

where $r(x, y)$ is the Radon-Nikodym derivative as defined in [29]

$$(2) \quad r(x, y) := \frac{\pi(dy)q(y, dx)}{\pi(dx)q(x, dy)} \quad \text{and} \quad \rho(x) := 1 - \int \min\{1, r(x, y)\} q(x, dy),$$

where q is the so-called proposal kernel (or proposal distribution). We follow the terminology of [3] and call this method the *marginal algorithm*.

In some situations, the marginal algorithm cannot be implemented due to the intractability of the distribution π . For example, assuming that π and q have densities (also denoted π and q) with respect to some σ -finite measure, it may be that π cannot be evaluated point-wise, and although $r(x, y)$ may be well defined theoretically, it cannot be evaluated either. However in some situations unbiased non-negative estimates $\hat{\pi}(x) = W_x \pi(x)$ may be available; that is, $W_x \sim Q_x(\cdot) \geq 0$ and $\mathbb{E}[W_x] = 1$ for any $x \in X$ (we will refer to W_x as a “weight” throughout the

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paper). A naive idea may be to use such estimates in place of the true values in order to compute the acceptance probability. A remarkable property is that such an algorithm is in fact correct [3]. This can be seen by consider the following probability distribution

$$(3) \quad \tilde{\pi}(\mathrm{d}x, \mathrm{d}w) := \pi(\mathrm{d}x)\pi_x(\mathrm{d}w) \quad \text{with} \quad \pi_x(\mathrm{d}w) := Q_x(\mathrm{d}w)w$$

on the product space $(\mathbf{X} \times \mathbf{W}, \mathcal{B}(\mathbf{X}) \times \mathcal{B}(\mathbf{W}))$ where \mathbf{W} is a Borel subset of \mathbb{R}_+ and $\mathcal{B}(\mathbf{W})$ are the Borel sets on \mathbf{W} . Here $\pi_x(\mathrm{d}w)$ is a probability measure for each $x \in \mathbf{X}$, and therefore π is a marginal distribution of $\tilde{\pi}$.

It is possible to implement a Metropolis-Hasting algorithm targeting $\tilde{\pi}(\mathrm{d}x, \mathrm{d}w)$ using a proposal kernel $\tilde{q}(x, w; \mathrm{d}y, \mathrm{d}u) := q(x, \mathrm{d}y)Q_y(\mathrm{d}u)$ by defining

$$(4) \quad \tilde{P}(x, w; \mathrm{d}y, \mathrm{d}u) := \min \left\{ 1, r(x, y) \frac{u}{w} \right\} q(x, \mathrm{d}y)Q_y(\mathrm{d}u) + \delta_{x,w}(\mathrm{d}y, \mathrm{d}u)\tilde{\rho}(x, w),$$

where the probability of rejection is given as

$$\tilde{\rho}(x, w) := 1 - \iint \min \left\{ 1, r(x, y) \frac{u}{w} \right\} q(x, \mathrm{d}y)Q_y(\mathrm{d}u).$$

This is the *pseudo-marginal algorithm* [3], which targets π marginally since it is a marginal distribution of $\tilde{\pi}$, and may be implemented in situations where the marginal algorithm may not. As a particular instance of the Metropolis-Hastings algorithm, the pseudo-marginal algorithm converges to $\tilde{\pi}$ under mild assumptions [e.g. 23], and although it may be seen as a “noisy” version of the marginal algorithm, it is exact since it allows us to target the distribution of interest π . The aim of this paper is to study some of the theoretical properties of such algorithms in terms of the properties of the weights and those of the marginal algorithm. More precisely we investigate the rate of convergence of the pseudo-marginal algorithm to equilibrium and characterise the approximation of the marginal algorithm by the pseudo-marginal algorithm in terms of the variability of their respective ergodic averages.

The apparently abstract structure of the pseudo-marginal algorithm is in fact shared by several practical algorithms which have recently been proposed in order to sample from intractable distributions. The distribution of w is most often implicit, as we illustrate now with one of the simplest examples. Assume for simplicity that the space \mathbf{X} is (a Borel subset of) \mathbb{R}^d and $\mathcal{B}(\mathbf{X})$ consists of the Borel subsets of \mathbf{X} and that both π and $q(x, \cdot)$ (for any $x \in \mathbf{X}$) have densities with respect to the Lebesgue measure. Consider a situation where the target density is of the form $\pi(x) = \int \pi(x, z)\mathrm{d}z$ where the integral cannot be computed analytically. One can suggest approximating this density with an importance sampling estimate of the integral,

$$(5) \quad W_x \pi(x) = \hat{\pi}(x) = \frac{1}{N} \sum_{n=1}^N \frac{\pi(x, Z_k)}{h_x(Z_k)} \quad Z_k \sim h_x(\cdot) \text{ independently,}$$

where h_x is a probability density for each $x \in \mathbf{X}$. Note that it is in fact possible to consider unbiased estimators up to a normalising constant since such a constant cancels in the acceptance ratio of the pseudo-marginal algorithm, and without loss of generality we will assume this constant to be equal to one throughout. This setting was considered by Beaumont in the seminal paper [7] and various extensions

proposed in [3]. There are more involved applications of this idea. In the context of state-space models, it has been shown in [1] that W_x can be obtained with a particle filter—resulting in “particle MCMC” algorithms. In [8] it was shown how exact sampling methods can be used to carry out inference in discretely observed diffusion models for which the transition probability is intractable. See also the discussion [17] on the connection with pseudo-marginal MCMC and approximate Bayesian computation.

We now summarise our main findings, which are of two different nature although some of their underpinnings and consequences related.

Rates of convergence. In previous work [3] it has been shown that a pseudo-marginal chain is uniformly ergodic whenever the marginal algorithm targeting $\pi(x)$ is uniformly ergodic and the weights are bounded uniformly in x . It was also shown that geometric ergodicity is not possible as soon as the weights W_x are unbounded on a set of positive π -probability. We extend the analysis of the convergence rates of the pseudo-marginal algorithms in several directions.

In Section 3, we show that if the marginal chain is geometric and the weights are bounded uniformly in x , then the pseudo-marginal chain is geometrically ergodic. Our proof relies on lower bounding the spectral gap (Propositions 7 and 9).

In most scenarios of interest, the support of the weight distributions is unbounded, implying that the corresponding pseudo-marginal algorithms cannot be geometric. We show that under various moment conditions on the weights, the pseudo-marginal algorithms have a specific sub-geometric rate of convergence. More precisely, in Section 5 assuming that the marginal algorithm is uniformly ergodic and the weight distributions are uniformly integrable we establish the existence of a sub-geometric drift condition towards a small set (Proposition 22) for an appropriate Lyapunov function. For example we show the existence of a polynomial drift condition (Corollary 23) when the weight distributions satisfy moment bounds. This together with an additional mild assumption allows us to establish sub-geometric rates of convergence.

In Section 6, we focus on the specific case where the marginal algorithm is the independent Metropolis-Hastings (IMH). We show that the existence of (not necessarily uniform) moment bounds for the weights lead to polynomial rates, while the existence of exponential moments leads to sub-exponential rates (Proposition 26 and its corollaries). In Section 7 we consider the popular random-walk Metropolis (RWM). Assuming standard tail conditions on π which ensure the geometric ergodicity of a RWM [13] and the existence of uniform moment bounds we show that the corresponding pseudo-marginal algorithm is polynomially ergodic (Theorem 32). We extend this result to non-uniform moment bounds case (i.e. allowing them to grow in the tail of π) in Theorem 40.

Asymptotic variance. It is natural to compare the asymptotic performance of ergodic averages obtained from a marginal algorithm and its pseudo-marginal counterpart. One can in fact ask a more general question of practical relevance. In practice, it is often possible to choose the weight distributions Q_x from a family $\{Q_x^N\}_{N \in \mathbb{N}}$ indexed by an accuracy parameter N , as for example in (5). In such situations $\pi_x^N(dw) = Q_x^N(dw)w$ converge weakly to $\delta_1(dw)$ as $N \rightarrow \infty$ and one

may wonder if the asymptotic variance of the corresponding ergodic averages converge to that of the marginal algorithm.

In Section 2 we first show that the pseudo-marginal and marginal algorithms are ordered both in terms of the mean acceptance probability (Corollary 3) and the asymptotic variance (Theorem 6). The latter result relies on a generalisation of the argument due to Peskun [24, 29]. This supports and generalises the empirical observation on toy examples that the pseudo-marginal algorithm cannot be more efficient than its marginal version.

When the weights are uniformly bounded in x , we start Section 4 with a simple upper bound on the asymptotic variance of the pseudo-marginal algorithm (Corollary 10) from which it is straightforward to deduce that it converges to that of the marginal when the weight upper bound goes to one. We generalise this result to the situation where the weights are unbounded, but $\pi_x^N(dw)$ converges weakly to $\delta_1(dw)$ as $N \rightarrow \infty$ (Theorem 15). We also show how the sub-geometric ergodicity results proved earlier are essential to establish the conditions of this theorem in practice (Proposition 19).

We conclude in Section 8 where we briefly discuss additional implications of our results such as the existence of central limit theorems, the possibility to compute quantitative expressions for the asymptotic variance and the analysis of generalisations of pseudo-marginal algorithms.

2. ORDERING OF THE MARGINAL AND PSEUDO-MARGINAL ALGORITHMS

We first introduce some standard notation related to probability measures and Markov transition probabilities. For Π a Markov kernel and μ a probability measure defined on some measurable space $(\mathbf{E}, \mathcal{B}(\mathbf{E}))$ and f a measurable real-valued function on \mathbf{E} , we let for any $x \in \mathbf{E}$, $\Pi^0 f(x) := f(x)$,

$$\mu(f) := \int f(x) \mu(dx) \quad \text{and} \quad \Pi^n f(x) := \int \Pi(x, dy) \Pi^{n-1} f(y) \text{ for } n \geq 1.$$

We will also denote the inner product between two real-valued functions f and g on \mathbf{E} as $\langle f, g \rangle_\mu := \int f(x) g(x) \mu(dx)$ and the associated norm $\|f\|_\mu := \langle f, f \rangle_\mu^{1/2}$.

We start by a simple lemma, which plays a key role in the ordering of the marginal and the pseudo-marginal algorithms.

Lemma 1. *For any $x, y \in \mathbf{X}$, we have*

$$\iint Q_x(dw) w Q_y(du) \min \left\{ 1, r(x, y) \frac{u}{w} \right\} \leq \min \{ 1, r(x, y) \}.$$

Proof. Notice that $t \mapsto \min\{1, t\}$ is a concave function. Therefore, one can apply Jensen's inequality, with the probability measure $Q_x(dw) w Q_y(du)$, to get the desired inequality. \square

In order to facilitate the comparison of P and \tilde{P} we follow [3] and introduce an auxiliary transition probability \bar{P} which is defined on the same space as the pseudo-marginal kernel \tilde{P} and is reversible with respect to $\tilde{\pi}$,

$$(6) \quad \bar{P}(x, w; dy, du) := q(x, dy) \pi_y(du) \min\{1, r(x, y)\} + \delta_{x,w}(dy, du) \rho(x).$$

Application of Lemma 1 leads to the generic result below, which in turn implies an order between the expected acceptance rates (Corollary 3) and the asymptotic variances (Theorem 6) of the marginal and pseudo-marginal algorithms.

Proposition 2. *Let $g : \mathsf{X}^2 \rightarrow [0, \infty)$ be a symmetric measurable function, that is such that $g(x, y) = g(y, x)$ for all $x, y \in \mathsf{X}$. Define*

$$\begin{aligned}\Delta_{\bar{P}}(g) &:= \int \tilde{\pi}(\mathrm{d}x, \mathrm{d}w) \int q(x, \mathrm{d}y) \pi_y(\mathrm{d}u) \min\{1, r(x, y)\} g(x, y) \\ \Delta_{\tilde{P}}(g) &:= \int \tilde{\pi}(\mathrm{d}x, \mathrm{d}w) \int q(x, \mathrm{d}y) Q_y(\mathrm{d}u) \min\left\{1, r(x, y) \frac{u}{w}\right\} g(x, y).\end{aligned}$$

Then we have $\Delta_{\bar{P}}(g) \geq \Delta_{\tilde{P}}(g)$ and whenever these quantities are finite,

$$\Delta_{\bar{P}}(g) - \Delta_{\tilde{P}}(g) \leq \int \pi(\mathrm{d}x) Q_x(\mathrm{d}w) |w - 1| \int q(x, \mathrm{d}y) \min\{1, r(x, y)\} g(x, y).$$

Proof. Denote $a(x, y, u, w) := \min\{1, r(x, y)\} - \min\left\{1, r(x, y) \frac{u}{w}\right\}$. Since $\int \pi_y(\mathrm{d}u) = 1 = \int Q_y(\mathrm{d}u)$, we may write for a bounded function g

$$\Delta_{\bar{P}}(g) - \Delta_{\tilde{P}}(g) = \int \pi(\mathrm{d}x) q(x, \mathrm{d}y) g(x, y) \int Q_x(\mathrm{d}w) w Q_y(\mathrm{d}u) a(x, y, u, w) \geq 0,$$

where the inequality is a consequence of Lemma 1. The general case follows by a truncation argument.

For the second bound, note that $\min\left\{1, r(x, y) \frac{u}{w}\right\} \geq \min\{1, r(x, y)\} \min\left\{1, \frac{u}{w}\right\}$ and $2 \min\{u, w\} = u + w - |u - w|$, and compute

$$\begin{aligned}\Delta_{\tilde{P}}(g) &\geq \int \pi(\mathrm{d}x) q(x, \mathrm{d}y) Q_x(\mathrm{d}w) Q_y(\mathrm{d}u) \min\{1, r(x, y)\} \min\{u, w\} g(x, y) \\ &= \Delta_{\bar{P}}(g) - \frac{1}{2} \int \pi(\mathrm{d}x) q(x, \mathrm{d}y) Q_x(\mathrm{d}w) Q_y(\mathrm{d}u) \min\{1, r(x, y)\} |u - w| g(x, y) \\ &\geq \Delta_{\bar{P}}(g) - \int \pi(\mathrm{d}x) Q_x(\mathrm{d}w) |1 - w| \int q(x, \mathrm{d}y) \min\{1, r(x, y)\} g(x, y),\end{aligned}$$

where the last inequality follows by the bound $|u - w| \leq |1 - u| + |1 - w|$, the symmetry of $g(x, y)$ and because

$$\pi(\mathrm{d}x) q(x, \mathrm{d}y) \min\{1, r(x, y)\} = \pi(\mathrm{d}y) q(y, \mathrm{d}x) \min\{1, r(y, x)\}. \quad \square$$

Corollary 3. *Let us denote the expected acceptance rates of the marginal and the pseudo-marginal algorithms as*

$$\begin{aligned}\alpha_P &:= \int \pi(\mathrm{d}x) \int q(x, \mathrm{d}y) \min\{1, r(x, y)\}, \\ \alpha_{\tilde{P}} &:= \int \tilde{\pi}(\mathrm{d}x, \mathrm{d}w) \int q(x, \mathrm{d}y) Q_y(\mathrm{d}u) \min\left\{1, r(x, y) \frac{u}{w}\right\},\end{aligned}$$

respectively. Then we have

$$0 \leq \alpha_P - \alpha_{\tilde{P}} \leq \int |w - 1| \pi(\mathrm{d}x) (1 - \rho(x)) Q_x(\mathrm{d}w) \leq \int |w - 1| \pi(\mathrm{d}x) Q_x(\mathrm{d}w).$$

Proof. Observe first that

$$\alpha_{\bar{P}} := \int \tilde{\pi}(\mathrm{d}x, \mathrm{d}w) \int q(x, \mathrm{d}y) Q_y(\mathrm{d}u) \min\{1, r(x, y)\} = \alpha_P.$$

Applying then Proposition 2 with $g \equiv 1$ implies

$$0 \leq \alpha_P - \alpha_{\bar{P}} \leq \int |w - 1| \pi(\mathrm{d}x) (1 - \rho(x)) Q_x(\mathrm{d}w).$$

The last inequality follows because $\rho(x) \in [0, 1]$ for all $x \in \mathbf{X}$. \square

Remark 4. Corollary 3 implies also the following bounds

$$\alpha_P - \alpha_{\bar{P}} \leq \begin{cases} \alpha_P \left(\sup_{x \in \mathbf{X}} \int Q_x(\mathrm{d}w) |1 - w| \right) \\ \alpha_P^{1/p} \left(\int \pi(\mathrm{d}x) Q_x(\mathrm{d}w) |1 - w|^q \right)^{1/q}, \end{cases}$$

where $p, q > 1$ with $1/p + 1/q = 1$.

We now define the notion of asymptotic variance for scaled ergodic averages of a Markov chain.

Definition 5. Let Π be a reversible Markov kernel with invariant distribution μ defined on some measurable space $(\mathbf{E}, \mathcal{B}(\mathbf{E}))$, and denote by $(X_k)_{k \geq 0}$ the corresponding Markov chain at stationarity, that is such that $X_0 \sim \mu$. Suppose $f : \mathbf{E} \rightarrow \mathbb{R}$ satisfies $\mu(f^2) < \infty$. The *asymptotic variance* of f under Π is defined as

$$(7) \quad \text{var}(f, \Pi) := \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \left(\sum_{k=1}^n f(X_k) - \mu(f) \right)^2 \in [0, \infty].$$

Whenever the *integrated autocorrelation time*

$$\tau(f, \Pi) := 1 + 2 \sum_{k=1}^{\infty} \frac{\mathbb{E}[f(X_0)f(X_k)] - \mu(f)^2}{\text{var}_{\mu}(f)} \quad \text{where} \quad \text{var}_{\mu}(f) := \mu(f - \mu(f))^2,$$

exists and is finite, then $\text{var}(f, \Pi) = \tau(f, \Pi) \text{var}_{\mu}(f) \in [0, \infty)$.

Lemma 47 in Appendix A shows that the limit in (7) always exists (but may be infinite) and proves the relation between $\tau(f, \Pi)$ and $\text{var}(f, \Pi)$. We now show that a pseudo-marginal algorithm is always dominated by its associated marginal algorithm in terms of asymptotic variance. The result can be regarded as an extension of Peskun's approach [24, 29]. We point out in the proof what makes the result not straightforward.

Theorem 6. Assume $f : \mathbf{X} \rightarrow \mathbb{R}$ satisfies $\pi(f^2) < \infty$. Denote $\text{var}(f, \tilde{P}) = \text{var}(\tilde{f}, \tilde{P})$ where $\tilde{f}(x, \cdot) \equiv f(x)$.

- (i) Then, $\text{var}(f, \tilde{P}) \geq \text{var}(f, P)$.
- (ii) More specifically,

$$\text{var}(f, \tilde{P}) \geq \text{var}(f, P) + \liminf_{\lambda \rightarrow 1^-} [\Delta_{\bar{P}}(g_{\lambda}) - \Delta_{\tilde{P}}(g_{\lambda})]$$

where $\Delta_{\bar{P}}(g_{\lambda})$ and $\Delta_{\tilde{P}}(g_{\lambda})$ are defined in Proposition 2 and $g_{\lambda}(x, y) := [\phi_{\lambda}(x) - \phi_{\lambda}(y)]^2$ with $\phi_{\lambda}(x) := \sum_{k=0}^{\infty} \lambda^k [P^k f(x) - \pi(f)]$ for $\lambda \in [0, 1)$.

Proof. Our proof is inspired by the proof of Tierney [29, Theorem 4] but we cannot use his argument directly because Proposition 2 does not apply to functions depending also on u and w . Observe first from the definition of \bar{P} that a Markov chain $(\bar{X}_n, \bar{W}_n)_{n \geq 0}$ with the kernel \bar{P} and with $(\bar{X}_0, \bar{W}_0) \sim \tilde{\pi}$ coincides marginally with the marginal chain, that is, $(X_n)_{n \geq 0}$ following P with $X_0 \sim \pi$ and $(\bar{X}_n)_{n \geq 0}$ have the same distribution. Therefore, $\text{var}(f, \bar{P}) = \text{var}(f, P)$. We denote

$$\bar{f}(x) := f(x) - \pi(f) \in L_0^2(\mathbf{X}, \pi) := \{f : \mathbf{X} \rightarrow \mathbb{R} : \pi(f) = 0, \pi(f^2) < \infty\},$$

and with a slight abuse of notation define $\bar{f}(x, w) := \bar{f}(x)$ for all $(x, w) \in \mathbf{X} \times \mathbf{W}$. Notice that $\bar{f} \in L_0^2(\mathbf{X} \times \mathbf{W}, \tilde{\pi})$. For $\lambda \in [0, 1)$, we define the auxiliary quantities

$$\text{var}_\lambda(\bar{f}, H) = \langle \bar{f}, (I - \lambda H)^{-1}(I + \lambda H)f \rangle_{\tilde{\pi}},$$

for any Markov kernel H reversible with respect to $\tilde{\pi}$, where I stands for the identity operator. We note that from Lemma 46 in Appendix A the quantity $\text{var}_\lambda(\bar{f}, H)$ is well-defined and that from Lemma 47, it is sufficient to show that $\text{var}_\lambda(\bar{f}, \bar{P}) \leq \text{var}_\lambda(\bar{f}, \tilde{P})$ holds for all $\lambda \in [0, 1)$ in order to establish (i).

Using the notation of Lemma 46 with $P_1 = \bar{P}$ and $P_2 = \tilde{P}$, we can write

$$\begin{aligned} \text{var}_\lambda(\bar{f}, \tilde{P}) - \text{var}_\lambda(\bar{f}, \bar{P}) &= \langle \bar{f}, A_\lambda(1)\bar{f} \rangle_{\tilde{\pi}} - \langle \bar{f}, A_\lambda(0)\bar{f} \rangle_{\tilde{\pi}} \\ &= \int_0^1 \langle \bar{f}, A'_\lambda(\beta)\bar{f} \rangle_{\tilde{\pi}} d\beta \\ (8) \quad &= \int_0^1 \int_0^\beta \langle \bar{f}, A''_\lambda(\gamma)\bar{f} \rangle_{\tilde{\pi}} d\gamma d\beta + \int_0^1 \langle \bar{f}, A'_\lambda(0)\bar{f} \rangle_{\tilde{\pi}} d\beta. \end{aligned}$$

Note that if \tilde{P} and \bar{P} would satisfy Peskun's order, then the second line is sufficient to conclude [29]. We show now that both terms on the right hand side of the last line are non-negative.

First observe that by Lemma 46,

$$\langle \bar{f}, A'_\lambda(0)\bar{f} \rangle_{\tilde{\pi}} = 2\lambda \langle \bar{f}, (I - \lambda\bar{P})^{-1}(\tilde{P} - \bar{P})(I - \lambda\bar{P})^{-1}\bar{f} \rangle_{\tilde{\pi}} = 2\lambda \langle \phi_\lambda, (\tilde{P} - \bar{P})\phi_\lambda \rangle_{\tilde{\pi}},$$

due to the reversibility of \bar{P} , where $\phi_\lambda := (I - \lambda\bar{P})^{-1}\bar{f} = \sum_{k=0}^\infty \lambda^k \bar{P}^k \bar{f}$ is well-defined by Lemma 46. We notice that $\phi_\lambda(x, w) = \phi_\lambda(x)$, and a straightforward calculation (cf. (9)) shows that

$$\begin{aligned} \langle \phi_\lambda, (\tilde{P} - \bar{P})\phi_\lambda \rangle_{\tilde{\pi}} &= \int \tilde{\pi}(dx, dw) \phi_\lambda(x) \phi_\lambda(y) (\tilde{P}(x, w; dy, du) - \bar{P}(x, w; dy, du)) \\ &= \frac{1}{2} \int (\phi_\lambda(x) - \phi_\lambda(y))^2 \tilde{\pi}(dx, dw) (\bar{P}(x, w; dy, du) - \tilde{P}(x, w; dy, du)) \\ &= \frac{1}{2} [\Delta_{\bar{P}}(g_\lambda) - \Delta_{\tilde{P}}(g_\lambda)], \end{aligned}$$

with $g_\lambda(x, y) = (\phi_\lambda(x) - \phi_\lambda(y))^2$, and Proposition 2 yields $\langle \bar{f}, A'_\lambda(0)\bar{f} \rangle_{\tilde{\pi}} \geq 0$. We therefore turn our attention to

$$\begin{aligned} \langle \bar{f}, A''_\lambda(\gamma)\bar{f} \rangle_{\tilde{\pi}} &= 4\lambda^2 \langle \bar{f}, (I - \lambda H_\gamma)^{-1}(\tilde{P} - \bar{P})(I - \lambda H_\gamma)^{-1}(\tilde{P} - \bar{P})(I - \lambda H_\gamma)^{-1}\bar{f} \rangle_{\tilde{\pi}} \\ &= 4\lambda^2 \langle \varphi, (I - \lambda H_\gamma)^{-1}\varphi \rangle_{\tilde{\pi}} \end{aligned}$$

where $\varphi := (\tilde{P} - \bar{P})(I - \lambda H_\gamma)^{-1} \bar{f}$, by the reversibility of \bar{P} and \tilde{P} and the interpolated kernel $H_\gamma = \bar{P} + \gamma(\tilde{P} - \bar{P})$. It is easy to check that $\varphi \in L_0^2(\mathbf{X} \times \mathbf{W}, \tilde{\pi})$, so we may conclude (i) by applying Lemma 47 implying $\langle \varphi, (I - \lambda H_\gamma)^{-1} \varphi \rangle_{\tilde{\pi}} \geq 0$.

The specific lower bound (ii) follows from (8) because the first term is always non-negative. \square

3. GEOMETRIC ERGODICITY WHEN THE MARGINAL ALGORITHM IS GEOMETRIC AND THE WEIGHTS BOUNDED

We consider now an order between the spectral gaps of the pseudo-marginal kernel \tilde{P} and the auxiliary kernel \bar{P} in (6). Then, particularly, we find that if w is always bounded from above by $\bar{w} \in [1, \infty)$, that is, $\mathbf{W} = (0, \bar{w}]$, and P has a non-zero spectral gap (i.e. P is geometrically ergodic; see [26, Proposition 2.1]), then \tilde{P} has a non-zero spectral gap as well. We will also examine the asymptotic variance constants using the spectral gap order.

Suppose $f : \mathbf{X} \times \mathbf{W} \rightarrow \mathbb{R}$ is integrable with respect to $\tilde{\pi}$. We denote in this section the function centred with respect to w as

$$\bar{f}(x, w) := f(x, w) - f_0(x) \quad \text{where} \quad f_0(x) := \pi_x(f(x, \cdot)) = \int_0^\infty f(x, w) \pi_x(dw).$$

The Dirichlet form related to a Markov kernel Π with invariant distribution μ and a function g is given as

$$(9) \quad \mathcal{E}_\Pi(g) := \langle g, (I - \Pi)g \rangle_\mu = \frac{1}{2} \int \mu(dx) \Pi(x, dy) [g(x) - g(y)]^2,$$

where I is an identity operator. The spectral gap is defined through

$$(10) \quad \text{Gap}(\Pi) := \inf_{g: \text{var}_\mu(g) > 0} \frac{\mathcal{E}_\Pi(g)}{\text{var}_\mu(g)} = \inf_{g: \mu(g)=0, \|g\|_\mu=1} \mathcal{E}_\Pi(g),$$

where $\text{var}_\mu(g)$ is given in Definition 5.

Proposition 7. *The spectral gap of \bar{P} defined in (6) satisfies*

$$\text{Gap}(P) \wedge (1 - \text{ess sup}_{x \in \mathbf{X}} \rho(x)) \leq \text{Gap}(\bar{P}) \leq \text{Gap}(P),$$

where the essential supremum is with respect to π .

Proof. Let $f : \mathbf{X} \times \mathbf{W} \rightarrow \mathbb{R}$ with $\tilde{\pi}(f) = 0$ and $\|f\|_{\tilde{\pi}} = 1$ and compute

$$\begin{aligned} \mathcal{E}_{\bar{P}}(f) - \mathcal{E}_P(f_0) &= \frac{1}{2} \int \pi(dx) \pi_x(dw) q(x, dy) \pi_y(du) \min\{1, r(x, y)\} \\ &\quad ([f(x, w) - f(y, u)]^2 - [f_0(x) - f_0(y)]^2) \\ &= \int \pi(dx) \pi_x(dw) q(x, dy) \min\{1, r(x, y)\} [f^2(x, w) - f_0^2(x)] \\ &= \int \pi(dx) \pi_x(dw) [f(x, w) - f_0(x)]^2 (1 - \rho(x)). \end{aligned}$$

In other words,

$$(11) \quad \mathcal{E}_{\bar{P}}(f) = \mathcal{E}_P(f_0) + \int \pi(dx) \pi_x(dw) (1 - \rho(x)) \bar{f}^2(x, w).$$

If $\text{var}_\pi(f_0) > 0$, then we have by (11)

$$(12) \quad \begin{aligned} \mathcal{E}_{\bar{P}}(f) &\geq \text{Gap}(P) \text{var}_\pi(f_0) + \int \pi(\mathrm{d}x) \pi_x(\mathrm{d}w) (1 - \rho(x)) \bar{f}^2(x, w) \\ &\geq \text{Gap}(P) (1 - \tilde{\pi}(\bar{f}^2)) + (1 - \text{ess sup}_{x \in \mathbf{X}} \rho(x)) \tilde{\pi}(\bar{f}^2), \end{aligned}$$

where we have used that $1 = \text{var}_{\tilde{\pi}}(f) = \text{var}_\pi(f_0) + \tilde{\pi}(\bar{f}^2)$ by the variance decomposition identity. We notice that (12) holds also when $\text{var}_\pi(f_0) = 0$. We conclude with the bound $\mathcal{E}_{\bar{P}}(f) \geq \text{Gap}(P) \wedge (1 - \text{ess sup}_{x \in \mathbf{X}} \rho(x))$ which holds for all $\|f\|_{\tilde{\pi}} = 1$ with $\tilde{\pi}(f) = 0$, implying the first inequality.

For the second inequality, note that if $f(x, w) = f_0(x)$ for all $(x, w) \in \mathbf{X} \times \mathbf{W}$, then $\pi(f_0) = 0$ and $\pi(f_0^2) = 1$. Consequently, $\mathcal{E}_{\bar{P}}(f) = \mathcal{E}_P(f_0)$. Therefore, $\text{Gap}(\bar{P}) \leq \text{Gap}(P)$. \square

Remark 8. In the case where π is not concentrated on points, that is, $\pi(\{x\}) = 0$ for all $x \in \mathbf{X}$, the statement of Proposition 7 simplifies to $\text{Gap}(\bar{P}) = \text{Gap}(P)$, because then $1 - \text{ess sup}_{x \in \mathbf{X}} \rho(x) \geq \text{Gap}(P)$ by Lemma 49 (ii) in Appendix B.

Proposition 9. *Suppose that there exists a constant $\bar{w} \in [1, \infty)$ such that*

$$(13) \quad Q_x([0, \bar{w}]) = 1 \quad \text{for } \pi\text{-almost every } x \in \mathbf{X}.$$

Then, the Dirichlet form of the pseudo-marginal algorithm satisfies

$$\mathcal{E}_{\bar{P}}(f) \geq \bar{w}^{-1} \mathcal{E}_P(f),$$

for any function with $\tilde{\pi}(f^2) < \infty$, implying $\text{Gap}(\tilde{P}) \geq \bar{w}^{-1} \text{Gap}(\bar{P})$.

Proof. Because $\min\{1, ab\} \geq \min\{1, a\} \min\{1, b\}$ for all $a, b \geq 0$, we have

$$\begin{aligned} 2\mathcal{E}_{\bar{P}}(f) &= \int \tilde{\pi}(\mathrm{d}x, \mathrm{d}w) q(x, \mathrm{d}y) Q_y(\mathrm{d}u) \min\left\{1, r(x, y) \frac{u}{w}\right\} [f(x, w) - f(y, u)]^2 \\ &\geq \int_{u>0} \tilde{\pi}(\mathrm{d}x, \mathrm{d}w) q(x, \mathrm{d}y) \pi_y(\mathrm{d}u) \min\{1, r(x, y)\} \min\left\{\frac{1}{u}, \frac{1}{w}\right\} [f(x, w) - f(y, u)]^2 \\ &\geq 2\bar{w}^{-1} \mathcal{E}_P(f). \end{aligned} \quad \square$$

Corollary 10. *Assume $\text{Gap}(P) > 0$ and there exists some $\bar{w} \in [1, \infty)$ such that (13) holds. Let $g : \mathbf{X} \rightarrow \mathbb{R}$ satisfying $\pi(g^2) < \infty$, then, the asymptotic variances (Definition 5) satisfy*

$$\text{var}(g, P) \leq \text{var}(g, \tilde{P}) \leq \bar{w} \text{var}(g, P) + (\bar{w} - 1) \text{var}_\pi(g).$$

where $\text{var}(g, \tilde{P}) := \text{var}(\tilde{g}, \tilde{P})$ with $\tilde{g}(x, \cdot) \equiv g(x)$.

Proof. Proposition 9 implies $\langle f, (I - \tilde{P})f \rangle_{\tilde{\pi}} \geq \langle f, \bar{w}^{-1}(I - \bar{P})f \rangle_{\tilde{\pi}}$ for all functions $\tilde{\pi}(f^2) < \infty$, and Lemma 48 in Appendix B implies

$$\langle \tilde{g}, (I - \tilde{P})^{-1} \tilde{g} \rangle_{\tilde{\pi}} \leq \bar{w} \langle \tilde{g}, (I - \bar{P})^{-1} \tilde{g} \rangle_{\tilde{\pi}}.$$

Now note that $\text{var}_{\tilde{\pi}}(\tilde{g}) = \text{var}_\pi(g)$ and $\text{var}(\tilde{g}, \bar{P}) = \text{var}(g, P)$ hold because \bar{P} and P coincide marginally; see the proof of Theorem 6. The above, together with Theorem 6, imply,

$$\text{var}_\pi(g) + \text{var}(g, P) \leq \text{var}_{\tilde{\pi}}(\tilde{g}) + \text{var}(\tilde{g}, \tilde{P}) \leq \bar{w} (\text{var}_{\tilde{\pi}}(\tilde{g}) + \text{var}(\tilde{g}, \bar{P})),$$

and allows us to conclude. \square

Remark 11. From the proof of Proposition 9, one observes that in fact

$$\text{Gap}(\tilde{P}) \geq \text{Gap}(\check{P}) \geq \bar{w}^{-1} \text{Gap}(\bar{P}),$$

where \tilde{P} is the Markov kernel with the proposal $q(x, dy)Q_y(du)$ and the acceptance probability $\min\{1, r(x, y)\} \min\{1, u/w\}$ reversible with respect to $\tilde{\pi}$. This implies, repeating the arguments in the proof of Corollary 10, that $\text{var}(f, \tilde{P}) \leq \text{var}(f, \check{P})$ for all $\tilde{\pi}(f^2) < \infty$.

Next we show that the boundedness of the support of the weight distributions Q_x for essentially all $x \in \mathbf{X}$ is a necessary condition for geometric ergodicity of the pseudo-marginal algorithm. The result is similar to Theorem 8 in [3], but its proof is different and the statement more explicit.

Proposition 12. *If the pseudo-marginal kernel \tilde{P} has a non-zero spectral gap, then there exists a function $\bar{w} : \mathbf{X} \rightarrow [1, \infty)$ such that $Q_x([0, \bar{w}(x)]) = 1$ for π -a.e. $x \in \mathbf{X}$.*

Proof. We proceed by contradiction. Assume that there exists a set $A \in \mathcal{B}(\mathbf{X})$ with $\pi(A) > 0$ such that $Q_x([0, \tilde{w}]) < 1$ for all $x \in A$ and all $\tilde{w} \in [1, \infty)$. Fix $\epsilon > 0$ and let $\tilde{w} : A \rightarrow [1, \infty)$ be a measurable function such that $1 - \tilde{\rho}(x, w) \leq \epsilon$ for all $x \in A$ and $w \geq \tilde{w}(x)$, and such that $\tilde{\pi}(\tilde{A}) \in (0, 1/2)$ where $\tilde{A} := \{(x, w) \in \mathbf{X} \times \mathbf{W} : x \in A, w \geq \tilde{w}(x)\}$. We now apply Lemma 49 (i) in Appendix B with the set \tilde{A} , to conclude that $\text{Gap}(\tilde{P}) \leq (1 + (1 - \tilde{\pi}(\tilde{A}))^{-1})\epsilon \leq 3\epsilon$. \square

4. CONVERGENCE OF THE ASYMPTOTIC VARIANCE

In standard applications of the pseudo-marginal algorithm, one typically selects Q_x from a family of possible proposal distributions Q_x^N indexed by some precision parameter N which reflects the concentration of W on 1. In most relevant scenarios we are aware of, $N \in \mathbb{N}$ corresponds to the number of samples, particles or iterates of an algorithm used to compute an unbiased estimator of the density value, as exemplified in (5). It should be clear that this is not a restriction. Hereafter, we denote the pseudo-marginal kernels and the invariant measures associated with Q_x^N as \tilde{P}_N and $\tilde{\pi}_N$, respectively.

It is easy to see that if for all $x \in \mathbf{X}$, $Q_x^N(dw)w \rightarrow \delta_1(dw)$ as $N \rightarrow \infty$ weakly, then $\tilde{\pi}_N(dx, dw) \rightarrow \pi(dx)\delta_1(dw)$ weakly, suggesting that a pseudo-marginal algorithm with invariant distribution $\tilde{\pi}_N$ may become similar to the marginal algorithm with invariant distribution π as $N \rightarrow \infty$. As pointed out earlier, whenever W_x is not bounded uniformly, a pseudo-marginal algorithm cannot be geometric, although its marginal algorithm may be. In fact it was shown in [3, Remark 1] that even in situations where the weights are uniformly bounded and the pseudo-marginal algorithm is uniformly geometric, increasing N may not improve the rate of convergence of the algorithm, i.e. there is not convergence in terms of rate of convergence.

In this section we however show that in many situations such a convergence takes place in terms of the asymptotic variance, or equivalently, the integrated autocorrelation time; see Definition 5. More precisely, we show here that under simple conditions $\text{var}(g, \tilde{P}_N) \rightarrow \text{var}(g, P)$ as $N \rightarrow \infty$. We start with a very simple result, which is a direct consequence of Corollary 10.

Proposition 13. *Suppose that the marginal kernel P has a non-zero spectral gap and the weight distributions are bounded uniformly in $x \in \mathbf{X}$ by $\bar{w}^N \in (1, \infty)$, that is, $Q_x^N([0, \bar{w}^N]) = 1$ for all $x \in \mathbf{X}$ and $N \geq N_0$ for some $N_0 \in \mathbb{N}$, and $\lim_{N \rightarrow \infty} \bar{w}^N = 1$. Then, $\lim_{N \rightarrow \infty} \text{var}(g, \tilde{P}_N) = \text{var}(g, P)$ for any $g : \mathbf{X} \rightarrow \mathbb{R}$ with $\pi(g^2) < \infty$.*

Proof. The result is direct consequence of Corollary 10. \square

We now extend this result to situations where the distributions $\{Q_x^N\}_{N \in \mathbb{N}}$ may have an unbounded support, and therefore $\{\tilde{P}_N\}_{N \in \mathbb{N}}$ may not be geometrically ergodic. We formulate our result in terms of the following technical condition assuming uniform convergence of the integrated autocorrelation series. We will return to this assumption towards the end of this section and show that it can be checked in practice with for example Lyapunov type drift conditions (see Proposition 19).

Condition 14. For $g : \mathbf{X} \rightarrow \mathbb{R}$, suppose that the integrated autocorrelation time $\tau(g, P)$ (Definition 5) is well-defined and finite. Denote by $(\tilde{X}_k^N)_{k \geq 0}$ the Markov chain with initial distribution $\tilde{\pi}_N$ and kernel \tilde{P}_N . Assume that there exists a constant $N_0 < \infty$ such that

$$\lim_{n \rightarrow \infty} \sup_{N \geq N_0} \left| \sum_{k=n}^{\infty} \mathbb{E}[\bar{g}(\tilde{X}_0^N) \bar{g}(\tilde{X}_k^N)] \right| = 0 \quad \text{where} \quad \bar{g} = g - \pi(g).$$

The main result of this section is

Theorem 15. *Assume that $g : \mathbf{X} \rightarrow \mathbb{R}$ satisfies $\pi(|g|^{2+\delta}) < \infty$ and Condition 14 holds for g . Suppose also that,*

$$(14) \quad \lim_{N \rightarrow \infty} \int Q_x^N(dw) |1 - w| = 0 \quad \text{for all } x \in \mathbf{X}.$$

Then, $\lim_{N \rightarrow \infty} \text{var}(g, \tilde{P}_N) = \text{var}(g, P)$.

Proof. If $\text{var}_{\pi}(g) = 0$, the claim is trivial. If $\text{var}_{\pi}(g) > 0$, our conditions imply that the autocorrelation times exist and are finite for both the marginal kernel P and the pseudo-marginal kernels \tilde{P}_N for $N \geq N_0$; this follows from the finiteness of them terms in the autocorrelation series ensured by the Cauchy-Schwartz inequality, and Condition 14. Therefore, without loss of generality, we prove the claim for autocorrelation times $\tau(g, \tilde{P}_N) \rightarrow \tau(g, P)$ for a function g with $\tilde{\pi}_N(g) = \pi(g) = 0$ and $\tilde{\pi}_N(g^2) = \pi(g^2) = 1$.

Consider the Markov kernels \bar{P}_N defined as in (6) with Q_x^N and $\pi_x^N(dw) := Q_x^N(dw)w$. Denote by $(\bar{X}_k^N, \bar{W}_k^N)_{k \geq 0}$ the corresponding stationary Markov chain with $(\bar{X}_0^N, \bar{W}_0^N) \sim \tilde{\pi}_N$. Denote similarly $(\tilde{X}_k^N, \tilde{W}_k^N)_{k \geq 0}$ the stationary Markov chain corresponding to the kernel \tilde{P}_N with $(\tilde{X}_0^N, \tilde{W}_0^N) \sim \tilde{\pi}_N$. Notice that \bar{P}_N and $\tilde{\pi}_N$ coincide marginally with P and π , respectively, that is $(\bar{X}_k^N)_{k \geq 0}$ has the same distribution as that of the stationary marginal chain $(X_k)_{k \geq 0}$ with kernel P and such that $X_0 \sim \pi$.

Choose $\epsilon \in (0, 1)$ and let $n_0 = n_0(\epsilon) < \infty$ be such that for all $N \geq N_0$

$$(15) \quad \left| \sum_{k=n_0}^{\infty} \mathbb{E}[g(\tilde{X}_0^N) g(\tilde{X}_k^N)] \right| \leq \epsilon \quad \text{and} \quad \left| \sum_{k=n_0}^{\infty} \mathbb{E}[g(X_0) g(X_k)] \right| \leq \epsilon,$$

where the existence of n_0 follows from Condition 14. We have for $N \geq N_0$

$$|\tau(g, P) - \tau(g, \tilde{P}_N)| \leq 4\epsilon + 2 \left| \sum_{k=1}^{n_0-1} \mathbb{E}[g(\tilde{X}_0^N)g(\tilde{X}_k^N)] - \mathbb{E}[g(\bar{X}_0)g(\bar{X}_k)] \right|.$$

In order to control the last term, we consider a coupling argument. Denote $q := (2 + \delta)/\delta \in (1, \infty)$. Lemma 16 applied with $\tilde{\epsilon} = \epsilon n_0^{-q-1}/2$ implies the existence of $N_1 < \infty$ and a set $\bar{C} \in \mathcal{B}(\mathbf{X}) \times \mathcal{B}(\mathbf{W})$ such that for all $N \geq N_1$,

$$\begin{aligned} \tilde{\pi}_N(\bar{C}^{\mathbb{L}}) &\leq \epsilon n_0^{-q-1}/2 \\ \|\tilde{P}_N(x, w; \cdot) - \bar{P}_N(x, w; \cdot)\| &\leq \epsilon n_0^{-q-1}/2 \quad \text{for all } (x, w) \in \bar{C}. \end{aligned}$$

Lemma 51 in Appendix C applied to $(\tilde{X}_k^N, \tilde{W}_k^N)_{0 \leq k \leq n_0-1}$ and $(\bar{X}_k^N, \bar{W}_k^N)_{0 \leq k \leq n_0-1}$ with the set \bar{C} shows that the laws of these processes, $\tilde{\mu}$ and $\bar{\mu}$ respectively, satisfy the following total variation inequality for all $N \geq N_1$,

$$\|\tilde{\mu} - \bar{\mu}\| \leq 2n_0 \tilde{\pi}_N(\bar{C}^{\mathbb{L}}) + n_0 \sup_{(x, w) \in \bar{C}} \|\tilde{P}^N(x, w; \cdot) - \bar{P}^N(x, w; \cdot)\| \leq 2\epsilon n_0^{-q}.$$

Therefore, for all $N \geq N_1$, there exists a probability space $(\bar{\Omega}_N, \bar{\mathbb{P}}_N, \bar{\mathcal{F}}_N)$ where both $(\tilde{X}_k^N, \tilde{W}_k^N)_{0 \leq k \leq n_0-1}$ and $(\bar{X}_k^N, \bar{W}_k^N)_{0 \leq k \leq n_0-1}$ are defined, and the set

$$\bar{A}_N := \{(\tilde{X}_k^N, \tilde{W}_k^N) \equiv (\bar{X}_k^N, \bar{W}_k^N), 0 \leq k \leq n_0 - 1\}$$

satisfies $\bar{\mathbb{P}}_N(\bar{A}_N^{\mathbb{L}}) = \frac{1}{2} \|\tilde{\mu} - \bar{\mu}\| \leq \epsilon n_0^{-q}$ [e.g. 18, Theorem 5.2]. Denote $p = 1 + \delta/2$, and note that $p^{-1} + q^{-1} = 1$. Now for $N \geq N_1$,

$$\begin{aligned} &\left| \sum_{k=1}^{n_0-1} \mathbb{E}[g(\tilde{X}_0^N)g(\tilde{X}_k^N)] - \mathbb{E}[g(\bar{X}_0^N)g(\bar{X}_k^N)] \right| \\ &= \left| \bar{\mathbb{E}}_N \left[\sum_{k=1}^{n_0-1} g(\tilde{X}_0^N)g(\tilde{X}_k^N) - g(\bar{X}_0^N)g(\bar{X}_k^N) \right] \right| \\ &\leq (\bar{\mathbb{P}}_N(\bar{A}_N^{\mathbb{L}}))^{1/q} \left\{ \left(\bar{\mathbb{E}}_N \left| \sum_{k=1}^{n_0-1} g(\tilde{X}_0^N)g(\tilde{X}_k^N) - g(\bar{X}_0^N)g(\bar{X}_k^N) \right|^p \right)^{1/p} \right\} \\ &\leq (\bar{\mathbb{P}}_N(\bar{A}_N^{\mathbb{L}}))^{1/q} (n_0 - 1) \max_{1 \leq k \leq n_0-1} (\mathbb{E}|g(\tilde{X}_0^N)g(\tilde{X}_k^N)|^p)^{1/p} + (\mathbb{E}|g(\bar{X}_0)g(\bar{X}_k)|^p)^{1/p} \\ &\leq 2\epsilon^{1/q} (\pi(|g|^{2+\delta}))^{1/(2p)}, \end{aligned}$$

by the Hölder, Minkowski and Cauchy-Schwarz inequalities. \square

Let μ_1 and μ_2 be two probability distributions on the space $(\mathbf{E}, \mathcal{B}(\mathbf{E}))$. We define the total variation distance

$$\|\mu_1 - \mu_2\| := \sup_{|f| \leq 1} |\mu_1(f) - \mu_2(f)| = 2 \sup_{0 \leq f \leq 1} |\mu_1(f) - \mu_2(f)| = 2 \sup_{A \in \mathcal{B}(\mathbf{E})} |\mu_1(A) - \mu_2(A)|.$$

Lemma 16. *Assume that (14) is satisfied. Then, for any $\tilde{\epsilon} > 0$ there exists a $N_1 < \infty$ and a set $\check{C} \in \mathcal{B}(\mathbf{X}) \times \mathcal{B}(\mathbf{W})$ such that for all $N \geq N_1$,*

$$\begin{aligned} \tilde{\pi}_N(\check{C}^{\mathbb{L}}) &\leq \tilde{\epsilon} \\ \|\tilde{P}_N(x, w; \cdot) - \bar{P}_N(x, w; \cdot)\| &\leq \tilde{\epsilon} \quad \text{for all } (x, w) \in \check{C}. \end{aligned}$$

Proof. Choose $\check{\epsilon} > 0$ and let $\bar{w} := 1 + \check{\epsilon}/8$. The dominated convergence theorem together with (14) implies for all $x \in \mathbf{X}$,

$$(16) \quad \lim_{N \rightarrow \infty} \int q(x, dy) Q_y^N(du) |1 - u| = 0$$

$$(17) \quad \lim_{N \rightarrow \infty} \pi_x^N([\bar{w}^{-1}, \bar{w}]) = 1.$$

By Egorov's theorem, there exists a set $C \in \mathcal{B}(\mathbf{X})$ such that $\pi(C^c) \leq \check{\epsilon}/2$ and the convergence in both (16) and (17) is uniform in x .

For any $x \in \mathbf{X}$, any $w > 0$ and any set $A \in \mathcal{B}(\mathbf{X}) \times \mathcal{B}(\mathbf{W})$,

$$\begin{aligned} & |\tilde{P}_N(x, w; A) - \bar{P}_N(x, w; A)| \\ & \leq 2 \int q(x, dy) Q_y^N(du) \left| \min\{1, r(x, y)\}u - \min\left\{1, r(x, y)\frac{u}{w}\right\} \right| \\ & \leq 2 \int q(x, dy) Q_y^N(du) \left[|1 - u| + \left| \min\{1, r(x, y)\} - \min\left\{1, r(x, y)\frac{u}{w}\right\} \right| \right] \\ & \leq 2 \int q(x, dy) Q_y^N(du) \left[|1 - u| + \left| 1 - \frac{u}{w} \right| \right] \\ & \leq 2 \left| 1 - \frac{1}{w} \right| + 4 \int q(x, dy) Q_y^N(du) |1 - u|, \end{aligned}$$

where the third inequality follows by Lemma 50 in Appendix B. Therefore, letting $\check{C} := C \times [\bar{w}^{-1}, \bar{w}]$, we can bound the total variation by

$$\sup_{(x, w) \in \check{C}} \|\tilde{P}_N(x, w; \cdot) - \bar{P}_N(x, w; \cdot)\| \leq \frac{\check{\epsilon}}{2} + 8 \sup_{x \in C} \int q(x, dy) Q_y^N(du) |1 - u|.$$

Because $\lim_{N \rightarrow \infty} \tilde{\pi}_N(\check{C}^c) = \pi(C^c)$, we may conclude by choosing $N_1 < \infty$ such that $\sup_{x \in C} \int q(x, dy) Q_y^N(du) |1 - u| \leq \check{\epsilon}/16$ and $\tilde{\pi}_N(\check{C}^c) \leq \check{\epsilon}$ for all $N \geq N_1$. \square

Remark 17. With additional assumptions in Condition 14 and (14) on the rates of convergence, one could obtain a rate of convergence in Theorem 15, that is find $\{r(n)\}_{n \in \mathbb{N}}$ such that

$$|\text{var}(g, \tilde{P}_N) - \text{var}(g, P)| \leq r(N) \rightarrow 0 \quad \text{as } N \rightarrow \infty,$$

by going through the proofs of Theorem 15 and Lemma 16.

We now provide sufficient conditions implying the conditions of Theorem 15. Condition 14 which essentially require quantitative bounds on the ergodic behaviour of the pseudo-marginal Markov chains. Our results rely on polynomial drift conditions which we establish for some standard algorithms in Sections 6 and 7. Weaker drift conditions can be shown to imply Condition 14 [e.g. 2, 4], but we do not detail this here in order to keep presentation simple.

Condition 18. There exists a function $V : \mathbf{X} \times \mathbf{W} \rightarrow [1, \infty)$, a set $C \in \mathcal{B}(\mathbf{X}) \times \mathcal{B}(\mathbf{W})$ with $\sup_{(x, w) \in C} V(x, w) < \infty$, constants $\alpha \in (0, 1]$, $b_V \in [0, \infty)$, $\epsilon_V \in (0, \infty)$ and $N_0 < \infty$, such that for all $N \geq N_0$

$$\tilde{P}_N V(x, w) \leq V(x, w) - \epsilon_V V^\alpha(x, w) + b_V \mathbb{I}\{(x, w) \in C\} \quad \text{for all } x \in \mathbf{X}, w \in \mathbf{W},$$

and for any $v \in [1, \infty)$, there exists probability measures $\{\nu^N\}_{N \geq N_0}$ and a constant $\epsilon_v \in (0, 1]$, such that for all $N \geq N_0$,

$$\tilde{P}_N(x, w; \cdot) \geq \epsilon_v \nu^N(\cdot) \quad \text{for all } (x, w) \in \mathbf{X} \times \mathbf{W} \text{ such that } V(x, w) \leq v.$$

Proposition 19. *Assume Condition 18 holds for the pseudo-marginal kernels \tilde{P}_N , and that for some $\lambda \in [0, 1)$ and $\kappa \in [0, 1)$,*

$$\begin{aligned} \|g\|_{V^{\alpha_{\kappa, \lambda}}} &= \sup_{(x, w) \in \mathbf{X} \times \mathbf{W}} \frac{|g(x)|}{V^{\alpha_{\kappa, \lambda}}(x, w)} < \infty \quad \text{where} \quad \alpha_{\kappa, \lambda} := \kappa \frac{\alpha(1 - \lambda)}{1 - \lambda\alpha} \\ \sup_{N \geq N_0} \tilde{\pi}_N(|g| + 1)V^{1 - \lambda\alpha} &< \infty, \end{aligned}$$

then Condition 14 holds.

Proof. From the assumptions, there exists a finite constant R such that for all $N \geq N_0$ and any $(x, w), (x', w') \in \mathbf{X} \times \mathbf{W}$,

$$\sum_{k \geq 0} r(k) |\tilde{P}_N^k g(x, w) - \tilde{P}_N^k g(x', w')| \leq R \|g\|_{V^{\alpha_{\kappa, \lambda}}} (V^{1 - \lambda\alpha}(x, w) + V^{1 - \lambda\alpha}(x', w') - 1),$$

where $r(k) := (k + 1)^{\frac{\alpha(1 - \lambda)(1 - \kappa)}{1 - \alpha}} \rightarrow \infty$ as $k \rightarrow \infty$ [4, Corollary 12]; see also [2, Proposition 3.4]. Note that we may write

$$\begin{aligned} |\mathbb{E}_{(x, w)}[\tilde{g}(\tilde{X}_k^N)]| &= \left| \tilde{P}_N^k g(x, w) - \int \tilde{\pi}_N(dy, du) \tilde{P}_N^k g(y, u) \right| \\ &\leq \int \tilde{\pi}_N(dy, du) |\tilde{P}_N^k g(x, w) - \tilde{P}_N^k g(y, u)|. \end{aligned}$$

Therefore, we have for $n \geq 0$

$$\begin{aligned} \left| \sum_{k=n}^{\infty} \mathbb{E}[\tilde{g}(\tilde{X}_0^N) \tilde{g}(\tilde{X}_k^N)] \right| &\leq \mathbb{E} \left[|\tilde{g}(\tilde{X}_0^N)| \sum_{k=n}^{\infty} |\mathbb{E}_{(\tilde{X}_0^N, \tilde{W}_0^N)}[\tilde{g}(\tilde{X}_k^N)]| \right] \\ &\leq \frac{\|g\|_{V^{\alpha_{\kappa, \lambda}}}}{r(n)} [\tilde{\pi}_N(|g|V^{1 - \lambda\alpha}) + \pi(|g|)\tilde{\pi}_N(V^{1 - \lambda\alpha})]. \quad \square \end{aligned}$$

5. SUB-GEOMETRIC ERGODICITY WITH UNIFORMLY ERGODIC MARGINAL ALGORITHM

We consider the situation where the marginal algorithm is uniformly ergodic. This often corresponds to scenarios where the state space $\mathbf{X} \subset \mathbb{R}^d$ is compact. It turns out that when the weight distributions $\{Q_x\}_{x \in \mathbf{X}}$ do not have bounded supports but are uniformly integrable, then the corresponding pseudo-marginal algorithm satisfies a sub-geometric drift condition towards a set $\mathbf{C} := \mathbf{X} \times (0, \bar{w}]$ for some $\bar{w} \in (1, \infty)$. Provided the marginal algorithm satisfies a practically mild additional condition in (18), the set \mathbf{C} is guaranteed to be small for the pseudo-marginal chain.

We start by assuming uniform integrability in a form given by the de la Vallée-Poussin theorem [e.g. 20, p.19 T22]. This allows us to quantify the strength of the sub-geometric drift in a convenient way, for example indicating that moment conditions imply polynomial drifts and consequently polynomial ergodicity.

Condition 20. There exists a non-decreasing convex function $\phi : [0, \infty) \rightarrow [1, \infty)$ satisfying

$$\liminf_{t \rightarrow \infty} \frac{\phi(t)}{t} = \infty \quad \text{and} \quad M_W := \sup_{x \in \mathbf{X}} \int \phi(w) Q_x(dw) < \infty.$$

We record a simple implication of Condition 20.

Lemma 21. *Assume Condition 20 holds. Then, there exists a function $a(w) : [0, \infty) \rightarrow [0, \infty)$ depending only on M_W and ϕ such that*

$$\sup_{y \in \mathbf{X}} \int_{u \geq w} u Q_y(du) \leq a(w) \quad \text{and} \quad \lim_{w \rightarrow \infty} a(w) = 0.$$

Proof. For any function $f : [0, \infty) \rightarrow [0, \infty)$ non-decreasing in $[w, \infty)$, we have

$$\int_{u \geq w} u Q_y(du) \leq \int u \frac{f(u)}{f(w)} Q_y(du).$$

The function $f(w) := \phi(w)/w$ is non-decreasing for w sufficiently large, therefore

$$\sup_{y \in \mathbf{X}} \int_{u \geq w} u Q_y(du) \leq M_W \frac{w}{\phi(w)} =: a(w) \xrightarrow{w \rightarrow \infty} 0. \quad \square$$

The next result establishes a drift away from large values of w for the pseudo-marginal chain, given that the marginal algorithm has an acceptance probability uniformly bounded away from zero. All uniformly (and geometrically) ergodic Markov chains satisfy this property [27, Proposition 5.1].

Proposition 22. *Suppose that the one-step expected acceptance probability of the marginal algorithm is bounded away from zero,*

$$\alpha_0 := \inf_{x \in \mathbf{X}} \int q(x, dy) \min\{1, r(x, y)\} > 0,$$

and Condition 20 holds.

Then, there exist constants $\delta > 0$ and $\bar{w} \in (1, \infty)$ such that

$$\tilde{P}V(x, w) \leq V(w) - \delta \frac{V(w)}{w} \mathbb{I}\{w \in [\bar{w}, \infty)\} + M_W \mathbb{I}\{w \in (0, \bar{w})\}.$$

where $V(x, w) := V(w) := \phi(w)$. The constants δ and \bar{w} can be chosen to depend on α_0 , ϕ and M_W only.

Proof. We can estimate

$$\begin{aligned} & \tilde{P}V(x, w) - V(w) \\ &= \iint q(x, dy) Q_y(du) \min\left\{1, r(x, y) \frac{u}{w}\right\} [\phi(u) - \phi(w)] \\ &\leq M_W - \iint q(x, dy) Q_y(du) \min\left\{1, r(x, y) \frac{u}{w}\right\} \mathbb{I}\{u < w\} [\phi(w) - \phi(u)] \\ &\leq M_W - \phi(w) \int q(x, dy) \min\{1, r(x, y)\} \int_{u < w/2} Q_y(du) \frac{u}{w} \left[1 - \frac{\phi(w/2)}{\phi(w)}\right], \end{aligned}$$

because $\min\{1, ab\} \geq \min\{1, a\} \min\{1, b\}$ for all $a, b \geq 0$. The convexity of ϕ implies $2\phi(w/2) \leq 1 + \phi(w)$, and therefore $\limsup_{w \rightarrow \infty} \phi(w/2)/\phi(w) \leq 1/2$. Because

$\int_{u < w/2} Q_y(du)u = 1 - \int_{u \geq w/2} Q_y(du)u$, we may apply Lemma 21. Now, for any $\delta_0 \in (0, \alpha_0/2)$, there exists $\bar{w}_0 \in (1, \infty)$ such that

$$\tilde{P}V(x, w) - V(w) \leq M_W - \delta_0 \frac{\phi(w)}{w} \quad \text{for all } w \in [\bar{w}_0, \infty).$$

The claim follows by taking $\bar{w} \in [\bar{w}_0, \infty)$ sufficiently large such that $\phi(w)/w > M_W/\delta_0$ for all $w \in [\bar{w}, \infty)$. \square

In practice, Condition 20 is often verified for moments, that is, $\phi(w) = w^\beta$. We record the following corollary to highlight the straightforward connection of β to the polynomial drift rate.

Corollary 23. *Suppose the conditions of Proposition 22 hold with $\phi(w) = w^\beta + 1$ for some $\beta > 1$. Then, the pseudo-marginal kernel satisfies the drift condition*

$$\tilde{P}V(x, w) \leq V(w) - \delta V^{\frac{\beta-1}{\beta}}(w) + b_V \mathbb{I}\{w \in (0, \bar{w})\},$$

where $V(w) := w^\beta + 1$ and $b_V := M_W + \delta V^{\frac{\beta-1}{\beta}}(\bar{w})$.

Proof. Follows from Proposition 22 observing that $w \leq (1+w^\beta)^{1/\beta} = V(w)^{1/\beta}$. \square

Proposition 22 and Corollary 23 establish a drift towards the set $\mathbf{X} \times (0, \bar{w}]$. We are left with showing that the set $(0, \bar{w}]$ is small.

Lemma 24. *Suppose there exists $\epsilon > 0$, an integer $n \in [1, \infty)$ and a probability measure ν on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ such that denoting the (sub-probability) kernel $P_{\text{acc}}(x, A) := \int_A q(x, dy) \min\{1, r(x, y)\}$ then for any $A \in \mathcal{B}(\mathbf{X})$,*

$$(18) \quad P_{\text{acc}}^n(x, A) \geq \epsilon \nu(A) \quad \text{for all } x \in \mathbf{X}.$$

Then, there exists $\bar{w}_0 \in (1, \infty)$, $\tilde{\epsilon} > 0$ and a probability measure $\tilde{\nu}$ on $(\mathbf{X} \times \mathbf{W}, \mathcal{B}(\mathbf{X}) \times \mathcal{B}(\mathbf{W}))$ such that for all $\bar{w} \in [\bar{w}_0, \infty)$,

$$\tilde{P}^n(x, w; \cdot) \geq \frac{\tilde{\epsilon}}{\bar{w}} \tilde{\nu}(\cdot) \quad \text{for all } (x, w) \in \mathbf{X} \times (0, \bar{w}].$$

Proof. Choose $\bar{w}_0 > 1$ sufficiently large so that $\epsilon_W := \inf_{y \in \mathbf{X}} \int Q_y(du) \min\{\bar{w}_0, u\} > 0$; such \bar{w}_0 exists due to Lemma 21 because

$$\int Q_y(du) \min\{\bar{w}_0, u\} \geq \int_{u < \bar{w}_0} Q_y(du)u = 1 - \int_{u \geq \bar{w}_0} Q_y(du)u.$$

We may write for $A \times B \in \mathcal{B}(\mathbf{X}) \times \mathcal{B}(\mathbf{W})$ and for $w \in (0, \bar{w}]$,

$$\begin{aligned} \tilde{P}(x, w; A, B) &\geq \int_A q(x, dy) \int_B Q_y(du) \min\left\{1, r(x, y) \frac{u}{w}\right\} \\ &\geq \int_A q(x, dy) \min\{1, r(x, y)\} \int_B Q_y(du) \min\left\{1, \frac{u}{\bar{w}}\right\} \\ &\geq \frac{1}{\bar{w}} \int P_{\text{acc}}(x, dy) \hat{P}_W(y, B), \end{aligned}$$

where $\hat{P}_W(y, B) = \int_B Q_y(du) \min\{\bar{w}_0, u\}$. We deduce recursively that

$$\begin{aligned} \tilde{P}^n(x, w; A, B) &\geq \frac{1}{\bar{w}^n} [\inf_{y \in \mathbf{X}} \hat{P}_W(y, (0, \bar{w}])]^{n-1} \int P_{\text{acc}}^n(x, dy) \hat{P}_W(y, B) \\ &\geq \frac{\epsilon_W^{n-1} \epsilon}{\bar{w}^n} \int_A \nu(dy) \hat{P}_W(y, B) =: \frac{\epsilon_W^{n-1} \epsilon}{\bar{w}^n} \tilde{\nu}_0(A \times B). \end{aligned}$$

We may take $\tilde{\nu}(A \times B) = \tilde{\nu}_0(A \times B) / \tilde{\nu}_0(\mathbf{X} \times \mathbf{W})$ and $\tilde{\epsilon} = \epsilon \tilde{\nu}_0(\mathbf{X} \times \mathbf{W}) > 0$. \square

Remark 25. The condition in (18) is more stringent than assuming P uniformly ergodic. However, it is the most common way to establish the n -step minorisation condition $P^n(x, \cdot) \geq \epsilon \nu(\cdot)$ in practice, which holds if and only if P is uniformly ergodic. In the case of a continuous state-space \mathbf{X} where $q(x, \{y\}) = 0$ and $\nu(\{y\}) = 0$ for all $x, y \in \mathbf{X}$ and $n = 1$, the condition in (18) is in fact equivalent to $P(x, \cdot) \geq \epsilon \nu(\cdot)$.

6. SUB-GEOMETRIC ERGODICITY WITH AN IMH AS MARGINAL ALGORITHM

The independent Metropolis-Hastings (IMH) algorithm is a specific case of the Metropolis-Hastings in (1) corresponding to a proposal $q(x, dy) = q(dy)$ for all $x \in \mathbf{X}$, such that $\pi \ll q$. The IMH can often be made uniformly ergodic by choosing q to have heavier tails than π , in which case the results in Section 5 are applicable. However, the uniformity assumptions required in Section 5 can be relaxed. Firstly, we may consider situations where the marginal IMH algorithm is not uniformly ergodic, but for example polynomially ergodic. Secondly, uniform integrability of the weight distributions $\{Q_x\}_{x \in \mathbf{X}}$ is not a requirement. The results of this section may be relevant for example to the analysis of the Particle IMH-EM algorithm presented in [5]. Our results are inspired by [14] establishing polynomial ergodicity and [11] exploring other sub-geometric rates for the IMH.

Proposition 26. *Denote $\mu(x) := \pi(dx)/q(dx)$. Suppose that there exists a strictly increasing $\phi : (0, \infty) \rightarrow [1, \infty)$ with $\liminf_{t \rightarrow \infty} \phi(t)/t > 0$, such that*

$$(19) \quad \int \tilde{\pi}(dx, dw) \phi(\mu(x)w) < \infty.$$

Then, there exists constants $M, c, \epsilon \in (0, \infty)$ and a probability measure ν on $(\mathbf{X} \times \mathbf{W}, \mathcal{B}(\mathbf{X}) \times \mathcal{B}(\mathbf{W}))$ such that for all $(x, w) \in \mathbf{X} \times \mathbf{W}$,

$$(20) \quad \tilde{P}V(x, w) \leq V(x, w) - c \frac{V(x, w)}{\phi^{-1}(V(x, w))}, \quad \mu(x)w > M$$

$$(21) \quad \tilde{P}(x, w; \cdot) \geq \epsilon \nu(\cdot), \quad \mu(x)w \leq M,$$

and $\nu(V) < \infty$, where $V(x, w) := \phi(\mu(x)w)$.

Proof. Denote $A_{x,w} := \{(y, u) \in \mathbf{X} \times \mathbf{W} : \frac{\mu(y)u}{\mu(x)w} \geq 1\}$ and $R_{x,w} := A_{x,w}^c$ and write

$$\begin{aligned} \tilde{P}V(x, w) &= \int_{A_{x,w}} \frac{V(y, u)}{\mu(y)} \pi(dy) Q_y(du) + \int_{R_{x,w}} \frac{V(y, u)}{\mu(x)} \frac{u}{w} \pi(dy) Q_y(du) \\ &\quad + V(x, w) \int_{R_{x,w}} \left(1 - \frac{\mu(y)u}{\mu(x)w}\right) q(dy) Q_y(du) \\ &\leq \frac{1}{\mu(x)w} \int \tilde{\pi}(dy, du) V(y, u) + V(x, w) \left(1 - \frac{\tilde{\pi}(R_{x,w})}{\phi^{-1}(V(x, w))}\right), \end{aligned}$$

because $\mu(y)u \geq \mu(x)w$ on $A_{x,w}$. The first term on the right vanishes and $\tilde{\pi}(R_{x,w}) \rightarrow 1$ as $\mu(x)w \rightarrow \infty$, and $\liminf_{u \rightarrow \infty} u/\phi^{-1}(u) > 0$, implying (20).

For (21), observe that for $\mu(x)w \leq M$,

$$\tilde{P}(x, w; B) \geq \int_B \min \left\{ \frac{1}{M}, \frac{1}{\mu(y)u} \right\} \tilde{\pi}(dy, du) =: \tilde{\nu}(B),$$

and we can take $\epsilon = \tilde{\nu}(\mathbf{X} \times \mathbf{W})$ and $\nu = \epsilon^{-1} \tilde{\nu}$, for which (19) implies $\nu(V) < \infty$. \square

Corollary 27. *If for some $\gamma > 0$,*

$$\int \tilde{\pi}(dx, dw) \exp [(\mu(x)w)^\gamma] < \infty,$$

then there exist constants $M, c, c_V \in (0, \infty)$ such that for $\mu(x)w \geq M$, we have the drift

$$\tilde{P}V(x, w) \leq V(x, w) - c\kappa(V(x, w)),$$

where $V(x, w) = \exp((\mu(x)w)^\gamma)$ and $\kappa(t) = t(\log t)^{-1/\gamma}$.

Proof. Proposition 26 applied with $\phi(t) = \exp(t^\gamma)$. \square

The type of drift in Corollary 27 implies faster than polynomial sub-exponential rates of convergence; see for example [10].

Corollary 28. *If for some $\beta \geq 1$*

$$\int \tilde{\pi}(dx, dw) (\mu(x)w)^\beta < \infty,$$

then there exist constants $M, c, c_V \in (0, \infty)$ such that for $\mu(x)w \geq M$, we have the polynomial drift

$$\tilde{P}V(x, w) \leq V(x, w) - cV^\alpha(x, w),$$

where $V(x, w) = (\mu(x)w)^\beta + 1$ and $\alpha = 1 - 1/\beta$.

Proof. Proposition 26 applied with $\phi(t) = t^\beta + 1$ implies that for all $\mu(x)w \geq M$,

$$\tilde{P}V(x, w) \leq V(x, w) - c \frac{V(x, w)}{(V(x, w) - 1)^{1/\beta}} \leq V(x, w) - cV^\alpha(x, w). \quad \square$$

The type of drift in Corollary 28 implies polynomial rates of convergence; see for example [14]. We notice that the result suggests that the pseudo-marginal algorithm may have a similar rate of convergence as that of the marginal algorithm. This is in contrast with the situation where the marginal algorithm is geometric and the weights unbounded.

7. POLYNOMIAL ERGODICITY WITH A RWM AS MARGINAL ALGORITHM

We consider next conditions to check a polynomial drift condition for the pseudo-marginal algorithm in the case where the marginal algorithm is a geometrically ergodic random-walk Metropolis (RWM), which targets a super-exponentially decaying target with regular contours [13]. The existence of such a drift, together with additional simple assumptions, implies polynomial rates of ergodicity, but also Condition 14 (essential for the convergence of the pseudo-marginal asymptotic variance to that of the marginal algorithm) and a central limit theorem for example.

Our results rely on moment conditions on the distributions $Q_x(dw)$. In Section 7.1 we assume the moments to be (essentially) uniform in x , while in Section 7.2 we consider the case where the behaviour of $Q_x(dw)$ can get worse as $|x| \rightarrow \infty$. It is possible to extend our results beyond the polynomial case. For example one may assume the existence of exponential moment conditions; see Remark 33. For the sake of clarity and brevity, we have opted to detail only the polynomial case here.

Throughout this section, we denote the regions of almost sure acceptance and possible rejection for the marginal and pseudo-marginal algorithms as

$$\begin{aligned} A_x &:= \left\{ z \in \mathbf{X} : \frac{\pi(x+z)}{\pi(x)} \geq 1 \right\}, & R_x &:= A_x^c, \\ A_{x,w} &:= \left\{ (z, u) \in \mathbf{X} \times \mathbf{W} : \frac{\pi(x+z)}{\pi(x)} \frac{u}{w} \geq 1 \right\}, & R_{x,w} &:= A_{x,w}^c, \end{aligned}$$

respectively, for all $x \in \mathbf{X}$ and $w \in \mathbf{W}$.

7.1. Uniform moment bounds. Consider the following moment condition on the distributions $\{Q_x\}_{x \in \mathbf{X}}$ where $\mathbf{X} = \mathbb{R}^d$.

Condition 29. Suppose there exist constants $\alpha' > 0$ and $\beta' > 1$ such that

$$(22) \quad M_W := \operatorname{ess\,sup}_{x \in \mathbf{X}} \int (w^{-\alpha'} \vee w^{\beta'}) Q_x(dw) < \infty,$$

where $a \vee b := \max\{a, b\}$ and the essential supremum is taken with respect to the Lebesgue measure.

We first establish the following simple lemma, used throughout this section, which guarantees that the moment condition above holds also for any intermediate exponents.

Lemma 30. *Given (22), then for all $\alpha \in [0, \alpha']$ and $\beta \in [0, \beta']$ and any $\gamma \in [-\alpha', \beta]$*

$$\operatorname{ess\,sup}_{x \in \mathbf{X}} \int (w^{-\alpha} \vee w^{\beta}) Q_x(dw) \leq M_W \quad \text{and} \quad \operatorname{ess\,sup}_{x \in \mathbf{X}} \int w^{\gamma} Q_x(dw) \leq M_W.$$

Proof. The first inequality follows by observing that $w^{-\alpha} \vee w^{\beta} \leq w^{-\alpha'} \vee w^{\beta'}$ for all $w > 0$. For the second one, suppose first that $\gamma \in [0, \beta']$. Then, $w^{\gamma} \leq w^{-\alpha'} \vee w^{\beta'}$, and the result follows from the first inequality. The case $\gamma \in [-\alpha', 0]$ is similar. \square

The following condition for the target density π was introduced in [13].

Condition 31. The target distribution π has a density with respect to the Lebesgue measure (also denoted π) which is continuously differentiable and supported on \mathbb{R}^d . The tails of π are super-exponentially decaying and have regular contours, that is,

$$\lim_{|x| \rightarrow \infty} \frac{x}{|x|} \cdot \nabla \log \pi(x) = -\infty \quad \text{and} \quad \limsup_{|x| \rightarrow \infty} \frac{x}{|x|} \cdot \frac{\nabla \pi(x)}{|\nabla \pi(x)|} < 0,$$

respectively, where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$. Moreover, the proposal distribution satisfies $q(x, A) = q(A - x) = \int_A q(y - x) dy$ with a symmetric density q bounded away from zero in some neighbourhood of the origin.

The following theorem establishes a polynomial drift given the conditions above.

Theorem 32. *Suppose \tilde{P} is a pseudo-marginal kernel with distributions $Q_x(dw)$ satisfying Condition 29 with some constants $\alpha' > 0$ and $\beta' > 1$, and that the corresponding marginal algorithm is a random walk Metropolis with invariant density π and proposal density q satisfying Condition 31.*

Define $V : \mathbf{X} \times \mathbf{W} \rightarrow [1, \infty)$ as follows

$$(23) \quad V(x, w) := c_\pi^\eta \pi^{-\eta}(x) (w^{-\alpha} \vee w^\beta) \quad \text{where} \quad c_\pi := \sup_{z \in \mathbf{X}} \pi(z),$$

for some constants $\eta \in (0, \alpha' \wedge 1)$, $\alpha \in (\eta, \alpha']$ and $\beta \in (0, \beta' - \eta)$.

Then, there exists constants $\bar{w}, M, b \in [1, \infty)$, $\underline{w} \in (0, 1]$ and $\delta_V > 0$ such that

$$(24) \quad \tilde{P}V(x, w) \leq \begin{cases} V(x, w) - \delta_V V^{\frac{\beta-1}{\beta}}(x, w), & \text{for all } (x, w) \notin \mathbf{C}, \\ b, & \text{for all } (x, w) \in \mathbf{C}, \end{cases}$$

where $\mathbf{C} := \{(x, w) \in \mathbf{X} \times \mathbf{W} : |x| \leq M, w \in [\underline{w}, \bar{w}]\}$.

Moreover, b, δ_V and \mathbf{C} depend only on the marginal algorithm, the constants α', β' and M_W in Condition 29 and the chosen constants α, β, η .

Proof. Let $\bar{w} \in [1, \infty)$ and $\delta'_V > 0$ be as in Lemma 36, so that $\tilde{P}V(x, w) \leq V(x, w) - \delta'_V V^{\frac{\beta-1}{\beta}}(x, w)$ for all $x \in \mathbf{X}$ and all $w \geq \bar{w}$. Then, apply Lemma 37 with the fixed value of \bar{w} to obtain a $M \in [1, \infty)$ and $\lambda \in [0, 1)$ such that

$$(25) \quad \tilde{P}V(x, w) \leq \lambda V(x, w) = V(x, w) - (1 - \lambda)V(x, w),$$

for all $w \in (0, \bar{w}]$ and $|x| \geq M$. Lemma 38 implies that (25) holds with all $x \in \mathbf{X}$ and $w \in (0, \underline{w}]$, with some $\lambda' \in [0, 1)$. Because $V \geq 1$, we conclude the claim for $(x, w) \notin \mathbf{C}$ with $\delta_V := \min\{\delta'_V, 1 - \lambda, 1 - \lambda'\}$. Lemma 38 implies the case $(x, w) \in \mathbf{C}$.

The dependence on b, δ_V and \mathbf{C} is clear from the proofs of Lemmas 37–38. \square

Remark 33. It is possible to generalise Theorem 32 for non-polynomial moments. Particularly, we may let $V(x, w) = c_\pi^\eta \pi^{-\eta}(x) \phi(w)$ where $\phi : (0, \infty) \rightarrow [1, \infty)$ is defined by

$$\phi(w) := \begin{cases} a(w), & w \in (0, 1] \\ b(w), & w \in (1, \infty), \end{cases}$$

with non-increasing $a : (0, 1] \rightarrow [1, \infty)$ and non-decreasing $b : (1, \infty) \rightarrow [1, \infty)$ satisfying

$$\lim_{w \rightarrow 0+} w^{-\eta} a(w) = \infty \quad \text{and} \quad \lim_{w \rightarrow \infty} b(w)/w = \infty,$$

and for some $\gamma > \eta$

$$\operatorname{ess\,sup}_{x \in \mathbf{X}} \int_0^1 a(w) Q_x(dw) < \infty \quad \text{and} \quad \operatorname{ess\,sup}_{x \in \mathbf{X}} \int_1^\infty b(w) w^\gamma Q_x(dw) < \infty.$$

Note that $a(w)$ and $b(w)$ must grow at least polynomially as $w \rightarrow 0+$ and $w \rightarrow \infty$, respectively. For example $b(w) = \exp(c_b w)$ allows one to establish the claim with the stronger drift condition

$$\tilde{P}V(x, w) \leq V(x, w) - \hat{\delta}_V \frac{V(x, w)}{\log \circ V(x, w)} \quad (x, w) \notin \mathbb{C},$$

instead of the polynomial drift in (23).

Remark 34. We believe that the negative moment condition and the presence of $w^{-\alpha}$ in the drift function are not necessary in order to establish polynomial ergodicity in general. It seems, however, difficult to establish a one-step drift condition without any control of the behaviour of the distributions Q_x near zero.

We first consider a simple result which is auxiliary to the other lemmas.

Lemma 35. *We have the following bounds for all $x, z \in \mathbf{X}$, $w > 0$, $\hat{\alpha} > 0$ and $\hat{\beta} > 1$.*

$$\begin{aligned} \text{(i)} \quad & \int \min \left\{ 1, \frac{u}{w} \right\} Q_x(du) \geq \frac{1}{w} \left(1 - \frac{1}{w^{\hat{\beta}-1}} \int u^{\hat{\beta}} Q_x(du) \right) \\ \text{(ii)} \quad & \int_{\{u: (z, u) \in A_{x, w}\}} Q_{x+z}(du) \geq 1 - w^{\hat{\alpha}} \left(\frac{\pi(x)}{\pi(x+z)} \right)^{\hat{\alpha}} \int u^{-\hat{\alpha}} Q_{x+z}(dz). \end{aligned}$$

Proof. The bound (i) follows by writing

$$\int \min \left\{ 1, \frac{u}{w} \right\} Q_x(du) = \frac{1}{w} \left(1 - \int_{u \geq w} (u - w) Q_x(du) \right) \geq \frac{1}{w} \left(1 - \int_{u \geq w} u Q_x(du) \right),$$

and using the estimate $\mathbb{I}\{u \geq w\} \leq (u/w)^{\hat{\beta}-1}$. For (ii), similarly

$$\int_{\{u: (z, u) \in A_{x, w}\}} Q_{x+z}(du) = 1 - \int_{\{u < w \frac{\pi(x)}{\pi(x+z)}\}} Q_{x+z}(du)$$

and use $\mathbb{I}\{u < w \frac{\pi(x)}{\pi(x+z)}\} \leq u^{-\hat{\alpha}} \left(w \frac{\pi(x)}{\pi(x+z)} \right)^{\hat{\alpha}}$. \square

We next consider the case where w is large, and establish a polynomial drift in this case.

Lemma 36. *Suppose the conditions of Theorem 32 hold. Then, there exist constants $\delta_V > 0$ and $\bar{w} \in [1, \infty)$ such that*

$$\tilde{P}V(x, w) \leq V(x, w) - \delta_V V^{\frac{\beta-1}{\beta}}(x, w) \quad \text{for all } x \in \mathbf{X} \text{ and } w \in [\bar{w}, \infty).$$

Proof. We may write for $w \geq \bar{w} \geq 1$

$$\frac{\tilde{P}V(x, w)}{V(x, w)} = \iint_{A_{x, w}} a_{x, w}(z, u) Q_{x+z}(du) q(dz) + \iint_{R_{x, w}} b_{x, w}(z, u) Q_{x+z}(du) q(dz),$$

where

$$(26) \quad a_{x,w}(z, u) := \left(\frac{\pi(x)}{\pi(x+z)} \right)^\eta \frac{u^{-\alpha} \vee u^\beta}{w^\beta}$$

$$(27) \quad b_{x,w}(z, u) := \left(\frac{\pi(x+z)}{\pi(x)} \right)^{1-\eta} \frac{u^{1-\alpha} \vee u^{1+\beta}}{w^{1+\beta}} + \left(1 - \frac{\pi(x+z)}{\pi(x)} \frac{u}{w} \right).$$

We now estimate both integrals by partitioning their integration domains into their intersections with the acceptance and the rejection sets of the marginal algorithm. For notational simplicity we denote $A_{x,w} \cap R_x = A_{x,w} \cap (R_x \times \mathbb{W})$ etc.

The bound for the first integral is straightforward,

$$\iint_{A_{x,w} \cap A_x} a_{x,w}(z, u) Q_{x+z}(du) q(dz) \leq \frac{M_W}{w^\beta}.$$

For the second one, observe that $1 \leq \left(\frac{\pi(x+z)}{\pi(x)} \frac{u}{w} \right)^\eta$ on $A_{x,w}$, implying

$$\begin{aligned} \iint_{A_{x,w} \cap R_x} a_{x,w}(z, u) Q_{x+z}(du) q(dz) \\ \leq \frac{1}{w^{\beta+\eta}} \iint_{A_{x,w} \cap R_x} u^{\eta-\alpha} \vee u^{\eta+\beta} Q_{x+z}(du) q(dz) \leq \frac{M_W}{w^{\beta+\eta}}, \end{aligned}$$

because $\beta + \eta \leq \beta'$. Similarly, because $\left(\frac{\pi(x+z)}{\pi(x)} \frac{u}{w} \right)^{1-\eta} \leq 1$ on $R_{x,w}$ we have

$$\begin{aligned} \iint_{R_{x,w}} \left(\frac{\pi(x+z)}{\pi(x)} \right)^{1-\eta} \frac{u^{1-\alpha} \vee u^{1+\beta}}{w^{1+\beta}} Q_{x+z}(du) q(dz) \\ \leq \frac{1}{w^{\beta+\eta}} \iint_{R_{x,w}} u^{\eta-\alpha} \vee u^{\eta+\beta} Q_{x+z}(du) q(dz) \leq \frac{M_W}{w^{\beta+\eta}}. \end{aligned}$$

We now turn to the the crucial remainder, which approaches unity as w grows.

$$\begin{aligned} \iint_{R_{x,w}} \left(1 - \frac{\pi(x+z)}{\pi(x)} \frac{u}{w} \right) Q_{x+z}(du) q(dz) \\ = 1 - \iint \min \left\{ 1, \frac{\pi(x+z)}{\pi(x)} \frac{u}{w} \right\} Q_{x+z}(du) q(dz) \\ \leq 1 - \iint \min \left\{ 1, \frac{\pi(x+z)}{\pi(x)} \right\} \min \left\{ 1, \frac{u}{w} \right\} Q_{x+z}(du) q(dz) \\ \leq 1 - \frac{\nu}{w} \int_{\{z : \frac{\pi(x+z)}{\pi(x)} \geq \nu\}} \left(1 - \frac{M_W}{w^{\beta'-1}} \right) q(dz), \end{aligned}$$

by Lemma 35 (i), where $\nu \in (0, 1)$. Lemma 53 (ii) in Appendix D implies the existence of a $\nu > 0$ such that $\inf_{x \in \mathbb{X}} q(\{z : \frac{\pi(x+z)}{\pi(x)} \geq \nu\}) > 0$. Therefore, there exists a $\nu_2 \in (0, \nu)$, such that whenever w is sufficiently large

$$\iint_{R_{x,w}} \left(1 - \frac{\pi(x+z)}{\pi(x)} \frac{u}{w} \right) Q_{x+z}(du) q(dz) \leq 1 - \frac{\nu_2}{w}.$$

Because $\beta > 1$, the terms of the order $w^{-\beta}$ or $w^{-\eta-\beta}$ vanish faster than w^{-1} when w increases. Consequently, we have for any $\nu_3 \in (0, \nu_2)$, by further assuming w

sufficiently large, that

$$\begin{aligned}\tilde{P}V(x, w) &\leq \left(1 - \frac{\nu_3}{w}\right) V(x, w) \\ &= V(x, w) - \nu_3 V^\kappa(x, w) (c_\pi \pi^{-\eta}(x))^{1-\kappa} \leq V(x, w) - \nu_3 V^\kappa(x, w),\end{aligned}$$

where $\kappa = \frac{\beta-1}{\beta} \in (0, 1)$. \square

Next we deduce that in the regime where w is bounded, we have a geometric drift.

Lemma 37. *Assume the conditions of Theorem 32 hold and let $\bar{w} \in [1, \infty)$. Then, there exist constants $\lambda \in [0, 1)$ and $M \in [1, \infty)$ such that*

$$\tilde{P}V(x, w) \leq \lambda V(x, w) \quad \text{for all } w \in (0, \bar{w}], |x| \geq M.$$

Proof. We may write

$$\frac{\tilde{P}V(x, w)}{V(x, w)} = 1 + \iint_{A_{x,w}} \hat{a}_{x,w}(z, u) Q_{x+z}(du) q(dz) + \iint_{R_{x,w}} \hat{b}_{x,w}(z, u) Q_{x+z}(du) q(dz),$$

where

$$(28) \quad \hat{a}_{x,w}(z, u) := \left(\frac{\pi(x)}{\pi(x+z)} \right)^\eta \frac{u^{-\alpha} \vee u^\beta}{w^{-\alpha} \vee w^\beta} - 1$$

$$(29) \quad \hat{b}_{x,w}(z, u) := \left(\frac{\pi(x+z)}{\pi(x)} \right)^{1-\eta} \frac{u}{w} \left[\frac{u^{-\alpha} \vee u^\beta}{w^{-\alpha} \vee w^\beta} - \left(\frac{\pi(x+z)}{\pi(x)} \right)^\eta \right].$$

Fix a constant $c > 1$ and define the following subsets $\bar{A}_x := \{z : \frac{\pi(x+z)}{\pi(x)} \geq c\}$ and $\bar{R}_x := \{z : \frac{\pi(x+z)}{\pi(x)} \leq \frac{1}{c}\}$, and the annulus between these two sets as $D_x := (\bar{A}_x \cup \bar{R}_x)^c = \{z : \frac{1}{c} < \frac{\pi(x+z)}{\pi(x)} < c\}$. Compute

$$(30) \quad \begin{aligned} \int_{D_x} \int_{(z,u) \in A_{x,w}} \hat{a}_{x,w}(z, u) Q_{x+z}(du) q(dz) \\ \leq \frac{c^\eta}{w^{-\alpha} \vee w^\beta} \int_{D_x} \int u^{-\alpha} \vee u^\beta Q_{x+z}(du) q(dz) \leq M_W c^\eta q(D_x), \end{aligned}$$

and

$$(31) \quad \begin{aligned} \int_{D_x} \int_{(z,u) \in R_{x,w}} \hat{b}_{x,w}(z, u) Q_{x+z}(du) q(dz) \\ \leq c^{1-\eta} \int_{D_x} \int_{u < cw} \left(\frac{u}{w} \right) \frac{u^{-\alpha} \vee u^\beta}{w^{-\alpha} \vee w^\beta} Q_{x+z}(du) q(dz) \leq M_W c^{2-\eta} q(D_x). \end{aligned}$$

Let then $\gamma \in (\eta, \alpha \wedge 1)$ such that $\gamma + \beta \leq \beta'$ and observe that $\left(\frac{\pi(x+z)}{\pi(x)} \frac{u}{w} \right)^{1-\gamma} \leq 1$ on $R_{x,w}$, and thereby

$$(32) \quad \begin{aligned} \int_{\bar{R}_x} \int_{(z,u) \in R_{x,w}} \hat{b}_{x,w}(z, u) Q_{x+z}(du) q(dz) \\ \leq \int_{\bar{R}_x} \int_{(z,u) \in R_{x,w}} \left(\frac{\pi(x+z)}{\pi(x)} \right)^{\gamma-\eta} \frac{u^{\gamma-\alpha} \vee u^{\gamma+\beta}}{w^{\gamma-\alpha} \vee w^{\gamma+\beta}} Q_{x+z}(du) q(dz) \\ \leq M_W c^{-(\gamma-\eta)}. \end{aligned}$$

Similarly, observe that $\left(\frac{\pi(x)}{\pi(x+z)}\frac{w}{u}\right)^\gamma \leq 1$ on $A_{x,w}$ and so

$$(33) \quad \begin{aligned} & \int_{\bar{R}_x} \int_{(z,u) \in A_{x,w}} \hat{a}_{x,w}(z,u) Q_{x+z}(du) q(dz) \\ & \leq \frac{c^{-(\gamma-\eta)}}{w^{\gamma-\alpha} \vee w^{\gamma+\beta}} \int_{\bar{R}_x} \int_{(z,u) \in A_{x,w}} u^{\gamma-\alpha} \vee u^{\gamma+\beta} Q_{x+z}(du) q(dz) \leq M_W c^{-(\gamma-\eta)}. \end{aligned}$$

It holds that $1 \leq \left(\frac{\pi(x)}{\pi(x+z)}\frac{w}{u}\right)$ on $R_{x,w}$, so we have

$$(34) \quad \begin{aligned} & \int_{\bar{A}_x} \int_{(z,u) \in R_{x,w}} \hat{b}_{x,w}(z,u) Q_{x+z}(du) q(dz) \\ & \leq \frac{1}{w^{-\alpha} \vee w^\beta} \int_{\bar{A}_x} \int_{(z,u) \in R_{x,w}} \left(\frac{\pi(x+z)}{\pi(x)}\right)^{-\eta} u^{-\alpha} \vee u^\beta Q_{x+z}(du) q(dz) \leq M_W c^{-\eta}. \end{aligned}$$

We are left with the term that will yield the geometric drift when $|x|$ is large,

$$\begin{aligned} & \int_{\bar{A}_x} \int_{(z,u) \in A_{x,w}} \hat{a}_{x,w}(z,u) Q_{x+z}(du) q(dz) \\ & \leq \frac{M_W c^{-\eta}}{w^{-\alpha} \vee w^\beta} - \int_{\bar{A}_x} q(dz) \int_{\{u:(z,u) \in A_{x,w}\}} Q_{x+z}(du) \\ & \leq M_W c^{-\eta} - q(\bar{A}_x) \left(1 - M_W \left(\frac{w}{c}\right)^{\alpha'}\right), \end{aligned}$$

by Lemma 35 (ii). Lemma 53 (iii) implies that $\delta := \liminf_{|x| \rightarrow \infty} q(\bar{A}_x) > 0$.

Let $\delta' \in (0, \delta)$ and fix $\epsilon > 0$ sufficiently small so that $6\epsilon - \delta(1 - \epsilon)^2 \leq -\delta'$, and let $c > 1$ be sufficiently large so that $M_W c^{-\eta} \leq \epsilon$ and $M_W \left(\frac{\bar{w}}{c}\right)^{\alpha'} \leq \epsilon$, and also that all (32), (33) and (34) are bounded by ϵ . Condition 31 implies that $\limsup_{|x| \rightarrow \infty} q(D_x) = 0$, and therefore there exists $M = M(c, \epsilon) > 0$ such that (30) + (31) $\leq \epsilon$ for all $|x| \geq M$. By possibly increasing the bound M to ensure that $q(\bar{A}_x) \geq \delta(1 - \epsilon)$, we have that the claim holds for all $|x| \geq M$ with the constant $\lambda = 1 - \delta'$. \square

We complete the results above by considering in particular very small values of w .

Lemma 38. *Suppose the conditions of Theorem 32 hold, and let $\bar{w}, M \in [1, \infty)$. Then, there exist constants $\underline{w} \in (0, 1)$, $\lambda \in (0, 1)$ and $b \in [1, \infty)$ such that*

$$(35) \quad \tilde{P}V(x, w) \leq b, \quad \text{for } |x| \leq M \text{ and } w \in [\underline{w}, \bar{w}]$$

$$(36) \quad \tilde{P}V(x, w) \leq \lambda V(x, w), \quad \text{for } x \in \mathbf{X} \text{ and } w \in (0, \underline{w}].$$

Proof. From the proof of Lemma 37, we have

$$\begin{aligned} \frac{\tilde{P}V(x, w)}{V(x, w)} &\leq 1 - \left(\iint_{A_{x,w}} Q_{x+z}(\mathrm{d}u)q(\mathrm{d}z) \right) + \tilde{a}_{x,w} + \tilde{b}_{x,w}, \quad \text{where} \\ \tilde{a}_{x,w} &:= \iint_{A_{x,w}} \left(\frac{\pi(x)}{\pi(x+z)} \right)^\eta \frac{u^{-\alpha} \vee u^\beta}{w^{-\alpha} \vee w^\beta} Q_{x+z}(\mathrm{d}u)q(\mathrm{d}z) \\ \tilde{b}_{x,w} &:= \iint_{R_{x,w}} \left(\frac{\pi(x+z)}{\pi(x)} \right)^{1-\eta} \frac{u}{w} \frac{u^{-\alpha} \vee u^\beta}{w^{-\alpha} \vee w^\beta} Q_{x+z}(\mathrm{d}u)q(\mathrm{d}z). \end{aligned}$$

Because $\left(\frac{\pi(x)}{\pi(x+z)} \frac{w}{u} \right)^\eta \leq 1$ on $A_{x,w}$ and $\left(\frac{\pi(x+z)}{\pi(x)} \frac{u}{w} \right)^{1-\eta} \leq 1$ on $R_{x,w}$,

$$\tilde{a}_{x,w} + \tilde{b}_{x,w} \leq \iint \frac{u^{\eta-\alpha} \vee u^{\eta+\beta}}{w^{\eta-\alpha} \vee w^{\eta+\beta}} Q_{x+z}(\mathrm{d}u)q(\mathrm{d}z) \leq \frac{M_W}{w^{\eta-\alpha} \vee w^{\eta+\beta}}.$$

This is enough to show that $\tilde{P}V(x, w) \leq (1 + M_W)V(x, w)$ for all $(x, w) \in \mathbf{X} \times \mathbf{W}$. Because V is bounded on $\{|x| \leq M, w \in [\underline{w}, \bar{w}]\}$, this implies the existence of $b = b(\bar{w}, \underline{w}, M) < \infty$ such that (35) holds.

Consider then (36). Let $\delta > 0$ be small enough so that $\inf_{x \in \mathbf{X}} q(A_x^\delta) \geq \epsilon > 0$, where $A_x^\delta := \{z : \frac{\pi(x+z)}{\pi(x)} \geq \delta\}$. Then,

$$\begin{aligned} \iint_{A_{x,w}} Q_{x+z}(\mathrm{d}u)q(\mathrm{d}z) &\geq \int_{A_x^\delta} q(\mathrm{d}z) \int_{\{u : (z,u) \in A_{x,w}\}} Q_{x+z}(\mathrm{d}u) \\ &\geq \int_{A_x^\delta} q(\mathrm{d}z) \left(1 - M_W \left(\frac{w}{\delta} \right)^{\alpha'} \right) \geq \frac{\epsilon}{2} \end{aligned}$$

for $w \in (0, \underline{w}]$ if \underline{w} is small enough. We may further decrease \underline{w} to ensure that $\tilde{a}_{x,w} + \tilde{b}_{x,w} \leq \epsilon/4$ for all $w \in (0, \bar{w}]$ and conclude (36) with $\lambda = 1 - \epsilon/4$. \square

7.2. Non-uniform moment bounds. We replace the uniform moments in Condition 29 here with the following assumption, which allows the moments of the distributions $\{Q_x\}_{x \in \mathbf{X}}$ to grow in the tails of π .

Condition 39. Let $\hat{w} : \mathbf{X} \rightarrow [1, \infty)$ be a function bounded on compact sets and tending to infinity as $|x| \rightarrow \infty$. Let $\psi : (0, \infty) \rightarrow [1, \infty)$ be a non-increasing function such that $\psi(t) \rightarrow \infty$ as $t \rightarrow 0$, and define $g(x) := \psi(\pi(x))$.

(i) There exist constants $\alpha' > 0$ and $\beta' > 1$ such that

$$\operatorname{ess\,sup}_{x \in \mathbf{X}} g^{-1}(x) \int u^{-\alpha'} \vee u^{\beta'} Q_x(\mathrm{d}u) \leq 1,$$

where the essential supremum is taken with respect to the Lebesgue measure.

(ii) There exist constants $\xi_w \in (0, \beta' - 1)$ and $\xi_\pi \in (0, \beta' - 1 - \xi_w)$,

$$(37) \quad \sup_{x \in \mathbf{X}} \frac{g(x)}{\hat{w}^{\xi_\pi}(x)} \sup_{z \in R_x} \left[\left(\frac{\pi(x+z)}{\pi(x)} \right)^{\xi_\pi} \frac{g(x+z)}{g(x)} \right] < \infty,$$

where $R_x := \{z : \frac{\pi(x+z)}{\pi(x)} < 1\}$ is the set of possible rejection for the marginal random-walk Metropolis algorithm.

(iii) For any constant $b > 1$, one must have

$$(38) \quad \sup_{x \in \mathbf{X}} \frac{M_W(b(|x| \vee 1))}{\hat{w}^{\xi_w}(x)} < \infty.$$

where $M_W : (0, \infty) \rightarrow (0, \infty)$ is defined as follows

$$M_W(r) := \operatorname{ess\,sup}_{|x| \leq r} \int u^{-\alpha'} \vee u^{\beta'} Q_x(du) \leq \operatorname{ess\,sup}_{|x| \leq r} g(x),$$

where the essential supremum is taken with respect to the Lebesgue measure.

Condition 39 may appear rather implicit and technical at first. However they, together with additional assumptions required in Theorem 40 below, are implied by the more meaningful assumptions in Condition 41 and Corollary 42, whose proof may help the reader gain some intuition.

Theorem 40. *Suppose \tilde{P} is a pseudo-marginal kernel corresponding to a random walk Metropolis with invariant density π and increment proposal density q satisfying Condition 31. Suppose Condition 39 holds with some $\alpha' > 0$ and $\beta' > 1$. Define $V : \mathbf{X} \times \mathbf{W} \rightarrow [1, \infty)$ as in (23), where the constant exponents satisfy*

$$\eta \in (0, \alpha' \wedge (\beta' - 1 - \xi_w) \wedge (1 - \xi_\pi)), \quad \alpha \in (\eta, \alpha'], \quad \beta \in (1 + \xi_w - \eta, \beta' - \eta)$$

and $\eta \leq (\beta' - \beta) \wedge 1 - \xi_\pi$.

Furthermore, suppose that there exists a function $c : \mathbf{X} \rightarrow [1, \infty)$ bounded on compact sets such that $\limsup_{|x| \rightarrow \infty} c(x)e^{-x} < \infty$ and

$$(39) \quad \limsup_{|x| \rightarrow \infty} \frac{\hat{w}^{\xi_\pi}(x)}{c^{\xi_c}(x)} = 0 \quad \text{where} \quad \xi_c \in (0, [(\beta' - \beta) \wedge \alpha \wedge 1] - \eta - \xi_\pi),$$

and that for any constant $b \in [1, \infty)$

$$(40) \quad \limsup_{|x| \rightarrow \infty} M_W(b|x|) \max \left\{ q(D_x), \frac{1}{c^\eta(x)}, \left(\frac{\hat{w}(x)}{c(x)} \right)^{\alpha'} \right\} = 0,$$

where $D_x := \{z : \frac{1}{c(x)} \leq \frac{\pi(x+z)}{\pi(x)} \leq c(x)\}$.

Then, there exist constants $\bar{w}, M, b \in [1, \infty)$, $\underline{w} \in (0, 1]$ and $\delta_V > 0$ such that the polynomial drift inequality (24) holds. Furthermore, the constants depend only on those of the marginal algorithm, the quantities $\alpha', \beta', \xi_w, \xi_\pi, \psi, \hat{w}$ involved in Condition 39, including the upper bounds in (37) and (38) (as a function of b), the chosen η, α, β, c and ξ_c , and the upper bounds (39) and (40).

Proof. The proof follows by applying Lemma 43 below and then Lemma 44 with c_w from Lemma 43, similarly to the proof of Theorem 32 by setting $\bar{w} := \sup_{|x| \leq M} \bar{w}(x)$, and observing that V is bounded on C . The dependence on the various quantities is clear from the proofs of Lemmas 43 and 44. \square

Before proving Lemmas 43 and 44, we give sufficient conditions to establish the conditions of Theorem 40.

Condition 41. Suppose Condition 31 holds and additionally there exists a constant $\rho > 1$ such that

$$\lim_{|x| \rightarrow \infty} \frac{x}{|x|^\rho} \cdot \nabla \log \pi(x) = -\infty.$$

Moreover, the increment proposal density q satisfies $q(x) \leq \bar{q}(|x|)$ for some bounded differentiable non-increasing function $\bar{q} : [0, \infty) \rightarrow [0, \infty)$ such that $\int_{\mathbf{X}} \bar{q}(|x|) dx < \infty$.

Corollary 42. *Suppose Condition 41 is satisfied, and that*

$$(41) \quad \int u^{-\alpha'} \vee u^{\beta'} Q_x(du) \leq c(1 \vee |x|)^{\rho'}$$

with some constants $c < \infty$ and $\rho' \in [0, \rho - 1)$. Then, for any

$$\eta \in (0, \alpha' \wedge (\beta' - 1) \wedge 1), \quad \alpha \in (\eta, \alpha'], \quad \beta \in (1 - \eta, \beta' - \eta)$$

and V defined in (23), the drift inequality (24) holds, with constants $\bar{w}, M, b \in [1, \infty)$, $\underline{w} \in (0, 1]$ and $\delta_V > 0$ only depending on the marginal algorithm and $\alpha', \beta', c, \rho'$ in (41) and the chosen α, β and η .

Proof. Choose the constants ξ_w and ξ_π sufficiently small so that the conditions on η, α and β in Theorem 40 are satisfied.

Fix a unit vector $u \in \mathbb{R}^d$ and define the function $\hat{\psi} : \mathbb{R}_+ \rightarrow [1, \infty)$ such that

$$\hat{\psi}(\pi(ru)) = \begin{cases} r, & r \geq R_0 \\ R_0, & r \in [0, R_0), \end{cases}$$

where $R_0 \in [1, \infty)$; this is always possible because the function $r \mapsto \pi(ru)$ is bounded away from zero on compact sets and monotone decreasing on the tail.

Define then $g(x) = c_g \hat{\psi}^{\rho'}(\pi(x))$, where the value of the constant $c_g \geq 1$ will be fixed later. In order to guarantee that Condition 39 (i) is satisfied for sufficiently large c_g , it is sufficient to show that

$$(42) \quad \limsup_{|x| \rightarrow \infty} g^{-1}(x) c |x|^{\rho'} < \infty.$$

Due to Lemma 52 in Appendix D, if $|x|$ is sufficiently large, then $g(x) = g(\zeta_x |x| u)$ for some $\zeta_x \in [b^{-1}, b]$, where $b \in [1, \infty)$ is a constant. Therefore, $g^{-1}(x) \leq (b^{-1} |x|)^{-\rho'}$, implying (42).

Define then $\hat{w}(x) := g^{\zeta_w}(x)$, where $\zeta_w = \xi_\pi^{-1} \vee \xi_w^{-1} \in (1, \infty)$. It is easy to check similarly to (42) that

$$\sup_{x \in \mathbf{X}} \frac{g(x)}{\hat{w}^{\xi_\pi}(x)} + \frac{M_W(b(|x| \vee 1))}{\hat{w}^{\xi_w}(x)} \leq 1 + \sup_{x \in \mathbf{X}} \frac{c'(b|x|)^{\rho'}}{\hat{w}^{\xi_w}(x)} < \infty.$$

It is also easy to check that

$$\sup_{z \in R_x} \left[\left(\frac{\pi(x+z)}{\pi(x)} \right)^{\xi_\pi} \frac{g(x+z)}{g(x)} \right] = \sup_{z \in R_x} \left[\left(\frac{\pi(x+z)}{\pi(x)} \right)^{\xi_\pi} \left(\frac{\hat{\psi}(\pi(x+z))}{\hat{\psi}(\pi(x))} \right)^{\rho'} \right]$$

is uniformly bounded in $x \in \mathbf{X}$. This is because it is sufficient to check the condition in the tails along a ray, that is, only for $z = r|x|$, $r \geq 1$. We conclude about the existence of a constant $c_g \in [1, \infty)$ such that Condition 39 holds.

Choose $\epsilon_c \in (0, \rho - 1 - \rho')$ and let $c(x) = \exp(|x|^{\epsilon_c})$. It is easy to check that there exists ξ_c such that (39) and (40) hold, using Lemma 54 in Appendix D to estimate $q(D_x)$. \square

We start by establishing a polynomial drift when w is large.

Lemma 43. *Suppose the conditions of Theorem 40 hold. Then, there exist constants $c_w \in [1, \infty)$ and $\delta_V > 0$ such that letting $\bar{w}(x) := c_w \hat{w}(x)$,*

$$\tilde{P}V(x, w) \leq V(x, w) - \delta_V V^{\frac{\beta-1}{\beta}}(x, w) \quad \text{for all } x \in \mathbb{R}^d \text{ and } w \in [\bar{w}(x), \infty).$$

Proof. We may write

$$\frac{\tilde{P}V(x, w)}{V(x, w)} = \iint_{A_{x,w}} a_{x,w}(z, u) Q_{x+z}(du) q(dz) + \iint_{R_{x,w}} b_{x,w}(z, u) Q_{x+z}(du) q(dz),$$

where $a_{x,w}$ and $b_{x,w}$ are defined in (26) and (27), respectively.

In what follows, for any $\nu > 0$, we will denote by $b_\nu \in (0, \infty)$ a constant chosen so that for all $x \in \mathbf{X}$, $\{x + z : \frac{\pi(x+z)}{\pi(x)} \geq \nu\} \subset B(0, b_\nu(|x| \vee 1))$; see Lemma 53 (i) in Appendix D. We also denote by $c \in [1, \infty)$ a constant whose value may change upon each appearance.

For the first integral, note that on $A_{x,w}$, $1 \leq \left(\frac{\pi(x+z)}{\pi(x)} \frac{u}{w}\right)^\eta$, so denoting $\delta := \eta + \beta - 1 - \xi_w > 0$, we have for $w \geq \hat{w}(x)$,

$$\begin{aligned} \iint_{A_{x,w} \cap A_x} a_{x,w}(z, u) Q_{x+z}(du) q(dz) &\leq \iint_{A_{x,w} \cap A_x} \frac{u^{\eta-\alpha} \vee u^{\eta+\beta}}{w^{\eta+\beta}} Q_{x+z}(du) q(dz) \\ &\leq \frac{1}{w^{1+\delta}} \left(\frac{M_W(b_1(|x| \vee 1))}{\hat{w}^{\xi_w}(x)} \right) \leq \frac{c}{w^{1+\delta}}, \end{aligned}$$

by Condition 39 (iii). For the second one, let $\gamma \in (\eta + \xi_\pi, \beta' - \beta]$, $\gamma < 1$, and observe that $1 \leq \left(\frac{\pi(x+z)}{\pi(x)} \frac{u}{w}\right)^\gamma$ on $A_{x,w}$, implying that with $\delta' := \gamma + \beta - 1 - \xi_\pi > 0$

$$\begin{aligned} \iint_{A_{x,w} \cap R_x} a_{x,w}(z, u) Q_{x+z}(du) q(dz) &\leq \int_{R_x} \left(\frac{\pi(x+z)}{\pi(x)} \right)^{\gamma-\eta} \frac{u^{\gamma-\alpha} \vee u^{\gamma+\beta}}{w^{\gamma+\beta}} Q_{x+z}(du) q(dz) \\ &\leq \frac{1}{w^{1+\delta'}} \int_{R_x} \left[\left(\frac{\pi(x+z)}{\pi(x)} \right)^{\xi_\pi} \frac{g(x+z)}{g(x)} \right] \frac{g(x)}{\hat{w}^{\xi_\pi}(x)} q(dz) \leq \frac{c}{w^{1+\delta'}}, \end{aligned}$$

whenever $w \geq \hat{w}(x)$, by Condition 39 (i) and (ii). Similarly, because $\left(\frac{\pi(x+z)}{\pi(x)} \frac{u}{w}\right)^{1-\gamma} \leq 1$ on $R_{x,w}$ we have for $w \geq \hat{w}(x)$

$$\begin{aligned} \iint_{R_{x,w} \cap R_x} \left(\frac{\pi(x+z)}{\pi(x)} \right)^{1-\eta} \frac{u^{1-\alpha} \vee u^{1+\beta}}{w^{1+\beta}} Q_{x+z}(du) q(dz) &\leq \frac{1}{w^{1+\delta'}} \int_{R_x} \left[\left(\frac{\pi(x+z)}{\pi(x)} \right)^{\xi_\pi} \frac{g(x+z)}{g(x)} \right] \frac{g(x)}{\hat{w}^{\xi_\pi}(x)} q(dz) \leq \frac{c}{w^{1+\delta'}}, \end{aligned}$$

and similarly, because $\left(\frac{\pi(x+z)}{\pi(x)} \frac{u}{w}\right)^{1-\eta} \leq 1$,

$$\begin{aligned} \iint_{R_{x,w} \cap A_x} \left(\frac{\pi(x+z)}{\pi(x)}\right)^{1-\eta} \frac{u^{1-\alpha} \vee u^{1+\beta}}{w^{1+\beta}} Q_{x+z}(du) q(dz) \\ \leq \frac{1}{w^{1+\delta}} \left(\frac{M_W(b_1(|x| \vee 1))}{\hat{w}^{\xi_w}(x)}\right) \leq \frac{c}{w^{1+\delta}}. \end{aligned}$$

As in the proof of Lemma 36, we may apply Lemma 35 (i) to obtain

$$\begin{aligned} \iint_{R_{x,w}} \left(1 - \frac{\pi(x+z)}{\pi(x)} \frac{u}{w}\right) Q_{x+z}(du) q(dz) \\ \leq 1 - \frac{\nu}{w} \int_{\{z : \frac{\pi(x+z)}{\pi(x)} \geq \nu\}} \left(1 - \frac{1}{w^{\beta'-1}} \int u^{\beta'} Q_{x+z}(du)\right) q(dz) \\ \leq 1 - \frac{\nu}{w} \int_{\{z : \frac{\pi(x+z)}{\pi(x)} \geq \nu\}} q(dz) \left(1 - \frac{1}{w^{\beta'-1-\xi_w}} \left(\frac{M_W(b_\nu(|x| \vee 1))}{\hat{w}^{\xi_w}(x)}\right)\right) \\ \leq 1 - \frac{\nu}{w} \int_{\{z : \frac{\pi(x+z)}{\pi(x)} \geq \nu\}} q(dz) \left(1 - \frac{c}{w^{\beta'-1-\xi_w}}\right). \end{aligned}$$

where we may choose $\nu \in (0, 1)$ such that $\inf_{x \in \mathbf{X}} q\left(z : \frac{\pi(x+z)}{\pi(x)} \geq \nu\right) > 0$; Lemma 53 (ii) ensures the existence of such a ν .

The terms of the order $w^{-(1+\delta)}$ or $w^{-(1+\delta')}$ vanish faster than w^{-1} as w increases. Consequently, we can choose $c_w \in [1, \infty)$ sufficiently large so that there exists a $\nu' > 0$ such that for all $x \in \mathbf{X}$ and $w \geq \bar{w}(x)$,

$$\begin{aligned} \tilde{P}V(x, w) &\leq \left(1 - \frac{\nu'}{w}\right) V(x, w) \\ &= V(x, w) - \delta_V V^\kappa(x, w) (c_\pi^\eta \pi^{-\eta}(x))^{1-\kappa} \leq V(x, w) - \delta_V V^\kappa(x, w), \end{aligned}$$

where $\kappa = \frac{\beta-1}{\beta} \in (0, 1)$. \square

Our last lemma concentrates on the cases where either $|x|$ is large and w bounded, or w is small.

Lemma 44. *Assume the conditions of Theorem 40 hold and let $\bar{w}(x) := c_w \hat{w}(x)$ for some constant $c_w \in [1, \infty)$. Then, there exist constants $\lambda \in (0, 1)$, $\underline{w} \in (0, 1)$, $M \in [1, \infty)$ and $c_V \in [1, \infty)$ such that*

$$(43) \quad \tilde{P}V(x, w) \leq \lambda V(x, w) \quad \text{for } |x| \geq M, w \in (\underline{w}, \bar{w}(x)]$$

$$(44) \quad \tilde{P}V(x, w) \leq \lambda V(x, w) \quad \text{for } x \in \mathbf{X}, w \in (0, \underline{w}]$$

$$(45) \quad \tilde{P}V(x, w) \leq c_V V(x, w) \quad \text{for } (x, w) \in \mathbf{X} \times \mathbf{W}.$$

Proof. We may write

$$\frac{\tilde{P}V(x, w)}{V(x, w)} = 1 + \iint_{A_{x,w}} \hat{a}_{x,w}(z, u) Q_{x+z}(du) q(dz) + \iint_{R_{x,w}} \hat{b}_{x,w}(z, u) Q_{x+z}(du) q(dz),$$

where $\hat{a}_{x,w}$ and $\hat{b}_{x,w}$ are given as in (28) and (29).

Define the subsets $\bar{A}_x := \{z : \frac{\pi(x+z)}{\pi(x)} \geq c(x)\}$, $\bar{R}_x := \{z : \frac{\pi(x+z)}{\pi(x)} \leq \frac{1}{c(x)}\}$ and $D_x := (\bar{A}_x \cup \bar{R}_x)^c = \{z : \frac{1}{c(x)} < \frac{\pi(x+z)}{\pi(x)} < c(x)\}$. Lemma 52 in Appendix D implies the existence of $b_1 \in [1, \infty)$ and $M_0 \in [1, \infty)$ such that $\bar{A}_x \cup D_x + x \subset B(0, b_1(|x| \vee 1))$ for all $x \in \mathbf{X}$. We decompose the two sums above into sub-sums on \bar{A}_x and \bar{R}_x , with again an obvious abuse of notation.

Observe that $1 \leq \left(\frac{\pi(x+z)}{\pi(x)} \frac{u}{w}\right)^\eta$ on $A_{x,w}$ and $\left(\frac{\pi(x+z)}{\pi(x)} \frac{u}{w}\right)^{1-\eta} \leq 1$ on $R_{x,w}$, implying

$$\begin{aligned}
 (46) \quad & \iint_{D_x \cap A_{x,w}} \hat{a}_{x,w}(z, u) Q_{x+z}(du) q(dz) + \iint_{D_x \cap R_{x,w}} \hat{b}_{x,w}(z, u) Q_{x+z}(du) q(dz) \\
 & \leq \int_{D_x} \int \frac{u^{\eta-\alpha} \vee u^{\eta+\beta}}{w^{\eta-\alpha} \vee w^{\eta+\beta}} Q_{x+z}(du) q(dz) \\
 & \leq \frac{M_W(b_1(|x| \vee 1)) q(D_x)}{w^{\eta-\alpha} \vee w^{\eta+\beta}},
 \end{aligned}$$

because $\eta \leq (\beta' - \beta) \wedge \alpha$.

Let then $\gamma := \eta + \xi_\pi + \xi_c < (\beta' - \beta) \wedge \alpha \wedge 1$ and notice again that $\left(\frac{\pi(x+z)}{\pi(x)} \frac{u}{w}\right)^{1-\gamma} \leq 1$ on $R_{x,w}$ and $\left(\frac{\pi(x)}{\pi(x+z)} \frac{w}{u}\right)^\gamma \leq 1$ on $A_{x,w}$. Therefore,

$$\begin{aligned}
 & \iint_{\bar{R}_x \cap A_{x,w}} \hat{a}_{x,w}(z, u) Q_{x+z}(du) q(dz) + \iint_{\bar{R}_x \cap R_{x,w}} \hat{b}_{x,w}(z, u) Q_{x+z}(du) q(dz) \\
 & \leq \int_{\bar{R}_x} \left(\frac{\pi(x+z)}{\pi(x)}\right)^{\gamma-\eta} \int \frac{u^{\gamma-\alpha} \vee u^{\gamma+\beta}}{w^{\gamma-\alpha} \vee w^{\gamma+\beta}} Q_{x+z}(du) q(dz) \\
 & \leq \frac{1}{w^{\gamma-\alpha} \vee w^{\gamma+\beta}} \left(\frac{\hat{w}^{\xi_\pi}(x)}{c^{\xi_c}(x)}\right) \int_{\bar{R}_x} \left[\left(\frac{\pi(x+z)}{\pi(x)}\right)^{\xi_\pi} \frac{g(x+z)}{g(x)}\right] \frac{g(x)}{\hat{w}^{\xi_\pi}(x)} q(dz),
 \end{aligned}$$

because $\frac{\pi(x+z)}{\pi(x)} \leq c^{-1}(x)$ on \bar{R}_x .

It holds that $1 \leq \left(\frac{\pi(x)}{\pi(x+z)} \frac{w}{u}\right)$ on $R_{x,w}$, so we have

$$\begin{aligned}
 & \int_{\bar{A}_x} \int_{(z,u) \in R_{x,w}} \hat{b}_{x,w}(z, u) Q_{x+z}(du) q(dz) \\
 & \leq \int_{\bar{A}_x} \left(\frac{\pi(x)}{\pi(x+z)}\right)^\eta \int_{(z,u) \in R_{x,w}} \frac{u^{-\alpha} \vee u^\beta}{w^{-\alpha} \vee w^\beta} Q_{x+z}(du) q(dz) \\
 & \leq \frac{M_W(b_1(|x| \vee 1)) c^{-\eta}(x)}{w^{-\alpha} \vee w^\beta}.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 & \int_{\bar{A}_x} \int_{(z,u) \in A_{x,w}} \hat{a}_{x,w}(z, u) Q_{x+z}(du) q(dz) \\
 & \leq \frac{M_W(b_1(|x| \vee 1)) c^{-\eta}(x)}{w^{-\alpha} \vee w^\beta} - \int_{\bar{A}_x \cap A_{x,w}} Q_{x+z}(du) q(dz).
 \end{aligned}$$

Now, by Lemma 35 (ii),

$$\begin{aligned} \int_{\bar{A}_x \cap A_{x,w}} Q_{x+z}(du)q(dz) &\geq \int \left(1 - \left(\frac{w}{c(x)}\right)^{\alpha'} \int u^{-\alpha'} Q_{x+z}(du)\right) q(dz) \\ &\geq q(\bar{A}_x) \left[1 - M_W(b_1(|x| \vee 1)) c_w^{\alpha'} \left(\frac{\hat{w}(x)}{c(x)}\right)^{\alpha'}\right], \end{aligned}$$

for all $w \in (0, c_w \hat{w}(x)]$.

Lemma 53 (iii) in Appendix D implies that $\delta := \liminf_{|x| \rightarrow \infty} q(\bar{A}_x) > 0$. Condition 39 together with (39) and (40) imply

$$(47) \quad \limsup_{|x| \rightarrow \infty} \frac{\tilde{P}V(x, w)}{V(x, w)} \leq 1 - \delta,$$

and we may conclude (43), by choosing any $\lambda \in (1 - \delta, 1)$ and finding a sufficiently large $M \in [1, \infty)$ such that the claim holds.

Consider then (44) and assume $|x| \leq M$. It is easy to verify that (47) holds with some $\delta' > 0$ when taking $\limsup_{w \rightarrow 0+}$ in the terms of the earlier decomposition. Finally, it is easy to check that (45) holds for $|x| \leq M$ similarly as (46), and the general case follows from (43) and Lemma 43. \square

8. CONCLUDING REMARKS

Our convergence rate results in Sections 3 and 5–7 allow one to establish central limit theorems. In the case where the pseudo-marginal kernel is geometrically ergodic, that is, \tilde{P} admits a non-zero spectral gap as discussed in Section 3, the central limit theorem (CLT) holds for all functions $f : \mathbf{X} \times \mathbf{W} \rightarrow \mathbb{R}$ such that $\tilde{\pi}(f^2) < \infty$ [26, Corollary 2.1]. Specifically, we have for all $g : \mathbf{X} \rightarrow \mathbb{R}$ with $\pi(g^2) < \infty$,

$$(48) \quad \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} [g(\tilde{X}_k) - \pi(g)] \xrightarrow{n \rightarrow \infty} \mathcal{N}(0, \text{var}(g, \tilde{P})) \quad \text{in distribution,}$$

where $\text{var}(g, \tilde{P}) \in [0, \infty)$ is given in Definition 5. It is possible to deduce upper bounds for the asymptotic variance $\text{var}(g, \tilde{P})$. Namely, Corollary 10 relates $\text{var}(g, \tilde{P})$ to $\text{var}(g, P)$, and from Lemma 47 (51),

$$\text{var}(g, P) \leq \frac{1 + (1 - \text{Gap}(P))}{1 - (1 - \text{Gap}(P))} \int e_{g - \pi(g), P}(dx) = \frac{2 - \text{Gap}(P)}{\text{Gap}(P)} \text{var}_{\pi}(g).$$

If the spectral gap of the marginal algorithm is not directly accessible, it can be bounded by the drift constants; see [6] and references therein, and also [16, Theorem 4.2 (ii)].

When \tilde{P} is polynomially ergodic, the class of functions g for which the CLT (48) holds is related to the exponent in the polynomial drift. For the convenience of the reader, we reformulate here a result due to Jarner and Roberts [14].

Theorem 45. *Suppose P is irreducible and aperiodic. Assume there exists $V : \mathbf{X} \times \mathbf{W} \rightarrow [1, \infty)$, $\alpha \in [0, 1)$, $b \in [0, \infty)$, $c \in (0, \infty)$, a petite set [e.g. 13, 21] $C \in \mathcal{B}(\mathbf{X}) \times \mathcal{B}(\mathbf{W})$ such that*

$$(49) \quad \tilde{P}V(x, w) \leq V(x, w) - cV^{\alpha}(x, w) + b\mathbb{I}\{(x, w) \in C\},$$

and that there exists $\eta \in [1 - \alpha, 1]$ with $\tilde{\pi}(V^{2\eta}) < \infty$ and

$$\sup_{(x,w) \in X \times W} \frac{|g(x)|}{V^{\alpha+\eta-1}(x,w)} < \infty,$$

then $\text{var}(g, \tilde{P}) \in [0, \infty)$ and the CLT (48) holds.

Theorem 45 is a restatement of [14, Theorem 4.2], because the pseudo-marginal kernel \tilde{P} is also irreducible and aperiodic if the marginal kernel P is. The asymptotic variance can also be upper bounded in the polynomial case; see [4] and [16, Theorem 5.2 (ii) and Remark 5.3]. It is also possible to deduce non-asymptotic mean square error bounds [16].

Finally some of our results apply directly to extensions of pseudo-marginal algorithms which directly make use of noisy estimates of the marginal's acceptance ratio [15, 22]. However despite some similitudes and simplifications, the corresponding processes differ fundamentally in that $\{X_k\}_{k \in \mathbb{N}}$ is a Markov chain in this case (as opposed to the pseudo-marginal scenario) and we are currently investigating these differences.

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APPENDIX A. LEMMAS FOR SECTION 2

In this section, $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ is a generic measurable space and μ is a probability measure on \mathbf{X} . We consider the Hilbert space

$$L_0^2(\mathbf{X}, \mu) := \{f : \mathbf{X} \rightarrow \mathbb{R} : \mu(f) = 0, \mu(f^2) < \infty\},$$

equipped with the inner product $\langle f, g \rangle_\mu := \int_{\mathbf{X}} f(x)g(x)\mu(dx)$. We denote the corresponding norm by $\|f\|_\mu := \langle f, f \rangle_\mu^{1/2}$ and the operator norm for $A : L_0^2(\mathbf{X}, \mu) \rightarrow L_0^2(\mathbf{X}, \mu)$ as $\|A\| := \sup\{\|Af\|_\mu : \|f\|_\mu = 1\}$.

Lemma 46. *Let P_1 and P_2 be two Markov kernels on space \mathbf{X} reversible with respect to μ , and define the family of interpolated kernels $H_\beta := P_1 + \beta(P_2 - P_1)$ for $\beta \in [0, 1]$ also reversible with respect to μ . Then,*

$$A_\lambda(\beta) := (I - \lambda H_\beta)^{-1}(I + \lambda H_\beta) = I + 2 \sum_{k=1}^{\infty} \lambda^k H_\beta^k$$

is a well-defined operator on $L_0^2(\mathbf{X}, \mu)$ for all $\lambda \in [0, 1)$ and $\beta \in [0, 1]$ as well as the right-hand derivatives, with limits taken with respect to the operator norm,

$$A'_\lambda(\beta) := \lim_{h \rightarrow 0+} h^{-1}(A_\lambda(\beta + h) - A_\lambda(\beta)) = 2\lambda(I - \lambda H_\beta)^{-1}(P_2 - P_1)(I - \lambda H_\beta)^{-1},$$

$$A''_\lambda(\beta) := \lim_{h \rightarrow 0+} h^{-1}(A'_\lambda(\beta + h) - A'_\lambda(\beta)) = 2\lambda(I - \lambda H_\beta)^{-1}(P_2 - P_1)A'_\lambda(\beta),$$

for all $\lambda \in [0, 1)$ and $\beta \in [0, 1)$.

Proof. The expression for $A_\lambda(\beta)$ follows by the Neumann series representation $(I - \lambda H_\beta)^{-1} = \sum_{k=0}^{\infty} (\lambda H_\beta)^k$ which is well-defined because $\|(\lambda H_\beta)^k\| \leq \lambda^k$. Let us check that $\beta \mapsto A_\lambda(\beta)$ is right differentiable on $[0, 1)$. Write for any $h \in (0, 1 - \beta)$

$$\begin{aligned} A_\lambda(\beta + h) - A_\lambda(\beta) &= (I - \lambda H_{\beta+h})^{-1} \lambda (H_{\beta+h} - H_\beta) + \Delta_{\lambda, \beta, h} (I + \lambda H_\beta) \\ &= \lambda h (I - \lambda H_\beta)^{-1} (P_2 - P_1) + \Delta_{\lambda, \beta, h} (I + \lambda H_\beta) \\ &\quad + \lambda h \Delta_{\lambda, \beta, h} (P_2 - P_1), \end{aligned}$$

where $\Delta_{\lambda, \beta, h} = (I - \lambda H_{\beta+h})^{-1} - (I - \lambda H_\beta)^{-1}$. The differentiability follows as soon as we show $\lim_{h \rightarrow 0+} h^{-1}(\Delta_{\lambda, \beta, h})$ exists. By the Neumann series representation, it is sufficient to show that $\lim_{h \rightarrow 0+} h^{-1}(H_{\beta+h}^k - H_\beta^k)$ exists for all $k \geq 0$. The claim is trivial with $k = 0$, and the cases $k \geq 1$ follow inductively by writing

$$\begin{aligned} H_{\beta+h}^k - H_\beta^k &= H_\beta^{k-1} (H_{\beta+h} - H_\beta) + (H_{\beta+h}^{k-1} - H_\beta^{k-1}) H_{\beta+h} \\ &= h H_\beta^{k-1} (P_2 - P_1) + (H_{\beta+h}^{k-1} - H_\beta^{k-1}) H_\beta \\ &\quad + h (H_{\beta+h}^{k-1} - H_\beta^{k-1}) (P_2 - P_1). \end{aligned}$$

Because $(I - \lambda H_\beta)A_\lambda(\beta) = I + \lambda H_\beta$, we may write

$$\begin{aligned} \lambda h (P_2 - P_1) &= (I - \lambda H_{\beta+h})A_\lambda(\beta + h) - (I - \lambda H_\beta)A_\lambda(\beta) \\ &= (I - \lambda H_{\beta+h})(A_\lambda(\beta + h) - A_\lambda(\beta)) - \lambda h (P_2 - P_1)A_\lambda(\beta), \end{aligned}$$

from which, multiplying with h^{-1} and taking limit as $h \rightarrow 0+$, we obtain

$$(50) \quad \lambda(P_2 - P_1) = (I - \lambda H_\beta)A'_\lambda(\beta) - \lambda(P_2 - P_1)A_\lambda(\beta).$$

The desired expression for $A'_\lambda(\beta)$ follows by observing that $I + A_\lambda(\beta) = 2(I - \lambda H_\beta)^{-1}$. Consider then $A'_\lambda(\beta)$. From (50), we obtain

$$(I - \lambda H_\beta)h^{-1}(A'_\lambda(\beta + h) - A'_\lambda(\beta)) = \lambda(P_2 - P_1)A'_\lambda(\beta + h) + \lambda(P_2 - P_1)h^{-1}(A_\lambda(\beta + h) - A_\lambda(\beta)).$$

We conclude by taking limits as $h \rightarrow 0+$. \square

Lemma 47. *Suppose Π is a Markov kernel reversible with respect to μ , and $(X_n)_{n \geq 0}$ is a Markov chain corresponding to the transition Π with $X_0 \sim \mu$. Then, for a function $f \in L_0^2(\mathbf{X}, \mu)$*

$$(51) \quad \text{var}(f, \Pi) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \left(\sum_{i=1}^n f(X_i) \right)^2 = \int \frac{1+x}{1-x} e_{f, \Pi}(dx) \in [0, \infty],$$

where $e_{f, \Pi}$ is a positive measure on $S \subset [-1, 1]$ satisfying $e_{f, \Pi}(S) = \|f\|_\mu^2$.

For any $f \in L_0^2(\mathbf{X}, \mu)$, whenever the series below is convergent, then the following equality holds,

$$(52) \quad \text{var}_\mu(f) + 2 \sum_{k=1}^{\infty} \mathbb{E}[f(X_0)f(X_k)] = \text{var}(f, \Pi) < \infty.$$

Moreover,

$$\text{var}_\lambda(f, \Pi) := \langle f, (I - \lambda \Pi)^{-1}(I + \lambda \Pi)f \rangle_\mu \in [0, \infty)$$

is well-defined for all $\lambda \in [0, 1)$, and satisfies $\lim_{\lambda \rightarrow 1-} \text{var}_\lambda(f, \Pi) = \text{var}(f, \Pi)$ and $\langle f, (I - \lambda \Pi)^{-1}f \rangle \geq 0$.

Proof. The reversibility of Π ensures that Π is a self-adjoint operator on $L_0^2(\mathbf{X}, \mu)$ with a spectral radius bounded by one. Therefore, by the spectral decomposition theorem, there exists a positive measure $e_{f, \Pi}$ on the Borel subsets of the spectrum $S \subset [-1, 1]$ and such that $\langle f, \Pi^k f \rangle = \int_S x^k e_{f, \Pi}(dx)$ for all $k \geq 0$ [e.g. 25, VII.2ff.]. Now, we may write

$$(53) \quad \begin{aligned} \frac{1}{n} \mathbb{E} \left(\sum_{i=1}^n f(X_i) \right)^2 &= \frac{1}{n} \left(n \mathbb{E}[f^2(X_0)] + 2 \sum_{i=1}^n \sum_{j < i} \mathbb{E}[f(X_0)f(X_{i-j})] \right) \\ &= \langle f, f \rangle_\mu + \frac{2}{n} \sum_{k=1}^n (n-k) \langle f, \Pi^k f \rangle_\mu \\ &= \int_S \left(1 + 2 \sum_{k=1}^n \frac{n-k}{n} x^k \right) e_{f, \Pi}(dx). \end{aligned}$$

Because $x(1-x)^{-1} = \sum_{k \geq 1} x^k$ for all $|x| < 1$, it is straightforward to verify by Kronecker's lemma that (51) holds. Similarly, whenever the sum in (52) is convergent, it is easy to see that the term (53) converges to (52).

The expression for $A_\lambda(1)$ in Lemma 46 allows us to write

$$\begin{aligned} \text{var}_\lambda(f, \Pi) &= \langle f, f \rangle_\mu + 2 \sum_{k=1}^{\infty} \lambda^k \langle f, \Pi^k f \rangle_\mu = \int \left(1 + 2 \sum_{k=1}^{\infty} (\lambda x)^k \right) e_{f, \Pi}(dx) \\ &= \int_{-1}^0 \frac{1 + \lambda x}{1 - \lambda x} e_{f, \Pi}(dx) + \int_0^1 \frac{1 + \lambda x}{1 - \lambda x} e_{f, \Pi}(dx) \in [0, \infty). \end{aligned}$$

We conclude that $\lim_{\lambda \rightarrow 1^-} \text{var}_\lambda(f, \Pi) = \text{var}(f, \Pi)$ by the monotone convergence theorem. For the last claim, we use the Neumann series definition of $(I - \lambda\Pi)^{-1}$,

$$\langle f, (I - \lambda\Pi)^{-1}f \rangle_\mu = \sum_{k=0}^{\infty} \lambda^k \langle f, \Pi^k f \rangle_\mu = \int_S \frac{1}{1 - \lambda x} e_{f, \Pi}(dx) \in [0, \infty). \quad \square$$

APPENDIX B. LEMMAS FOR SECTION 3

We include the following result for the sake of self-containedness; the idea of the proof was pointed out also in [9, Theorem A.2].

Lemma 48. *Let A and B be self-adjoint operators on a Hilbert space \mathcal{H} satisfying $0 \leq \langle f, Af \rangle \leq \langle f, Bf \rangle$ for all $f \in \mathcal{H}$, and the inverses A^{-1} and B^{-1} exist. Then, $0 \leq \langle f, B^{-1}f \rangle \leq \langle f, A^{-1}f \rangle$ for all $f \in \mathcal{H}$.*

Proof. The claim follows easily as soon as we prove $\langle f, A^{-1}f \rangle = \sup_{g \in \mathcal{H}} [2\langle g, f \rangle - \langle g, Ag \rangle]$. This identity follows from

$$\begin{aligned} \langle f, A^{-1}f \rangle - 2\langle g, f \rangle + \langle g, Ag \rangle &= \langle f - Ag, A^{-1}f \rangle + \langle g, Ag - f \rangle \\ &= \langle A^{-1}f - g, A(A^{-1}f - g) \rangle \geq 0, \end{aligned}$$

and because the supremum is attained with $g = A^{-1}f$. \square

Lemma 49. *Suppose P is a Metropolis-Hastings kernel given in (1) and $\rho(x)$ is given in (2). Then, the spectral gap of P defined in (10) satisfies*

(i) *for any set $A \in \mathcal{B}(\mathbf{X})$ with $\pi(A) \in (0, 1)$,*

$$\text{Gap}(P) \leq (1 - \pi(A))^{-1} (1 - \inf_{x \in A} \rho(x)),$$

(ii) *if π does not have point masses, that is, $\pi(\{x\}) = 0$ for all $x \in \mathbf{X}$, then*

$$\text{Gap}(P) \leq 1 - \rho(x) \quad \text{for } \pi\text{-almost every } x \in \mathbf{X}.$$

Proof. We first check (i). Denote $p = \mathbb{P}(A) \in (0, 1)$ and define $f(x) = a\mathbb{I}\{x \in A\} - b\mathbb{I}\{x \notin A\}$ where the constants $a, b \in (0, \infty)$ are chosen so that $\pi(f) = ap - b(1 - p) = 0$ and $\pi(f^2) = a^2p + b^2(1 - p) = 1$. We may compute

$$\begin{aligned} \mathcal{E}_P(f) &= \frac{1}{2} \int \pi(dx) q(x, dy) \min\{1, r(x, y)\} [f(x) - f(y)]^2 \\ &= (a + b)^2 \int_A \pi(dx) \int_{A^c} q(x, dy) \min\{1, r(x, y)\} \\ &\leq (a + b)^2 \int_A \pi(dx) (1 - \rho(x)) \leq (a + b)^2 p (1 - \inf_{x \in A} \rho(x)). \end{aligned}$$

Now, according to our choice of a and b ,

$$(a + b)^2 p = (1 - b^2(1 - p)) + 2b^2(1 - p) + b^2p = 1 + b^2 = (1 - p)^{-1}.$$

Consider then (ii). The case $\text{Gap}(P) = 0$ is trivial, so assume $\text{Gap}(P) > 0$ and assume the claim does not hold. Then, there exists an $\epsilon > 0$ and a set $A \in \mathcal{B}(\mathbf{X})$ with $p := \mathbb{P}(A) \in (0, 1)$ such that $1 - \rho(x) \leq \text{Gap}(P) - \epsilon$ for all $x \in A$. From (i), $\text{Gap}(P) \leq (1 - p)^{-1}(\text{Gap}(P) - \epsilon)$. Because π is not concentrated on points, we may choose p as small as we want, which leads to a contradiction. \square

Lemma 50. *Let $a, b \geq 0$. Then,*

$$(54) \quad |\min\{1, ab\} - \min\{1, a\}| \leq \min\{1, a\}|1 - b|.$$

Proof. If $a \leq 1$ and $ab \leq 1$, $a > 1$ and $ab > 1$, or either $a = 0$ or $b = 0$, then (54) is immediate. If $0 < a \leq 1$ and $ab > 1$, then $b > a^{-1} \geq 1$ and

$$|\min\{1, ab\} - \min\{1, a\}| = a|a^{-1} - 1| \leq a|b - 1| = \min\{1, a\}|1 - b|.$$

If $a > 1$ and $ab \leq 1$, then $b \leq a^{-1} < 1$ and (54) is established by

$$|\min\{1, ab\} - \min\{1, a\}| = 1 - ab \leq 1 - b = \min\{1, a\}|1 - b|. \quad \square$$

APPENDIX C. LEMMAS FOR SECTION 4

Lemma 51. *Suppose $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ are Markov chains on a common state space $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ with kernels P and Q , and initial distributions π and ϖ , respectively, which are invariant such that $\pi P = \pi$ and $\varpi Q = \varpi$. Then, the distributions of X and Y denoted as μ_X and μ_Y satisfy the following inequality for any $C \in \mathcal{B}(\mathbf{X})$,*

$$\|\mu_X - \mu_Y\| \leq \|\pi - \varpi\| + 2(n-1)\pi(C^{\mathbb{L}}) + (n-1) \sup_{x \in C} \|P(x, \cdot) - Q(x, \cdot)\|,$$

where $\|\mu_X - \mu_Y\| := \sup_{|f| \leq 1} |\mu_X(f) - \mu_Y(f)|$ denotes the total variation.

Proof. Let $A \in \mathcal{B}(\mathbf{X})$. We shall use the shorthand notation $x = x_{1:n} = (x_1, \dots, x_n)$ and denote $g_P^{(1:n)}(x) = \mathbb{I}\{x \in A\}$ and

$$g_P^{(1:k)}(x_{1:k}) := \int P(x_k, dx_{k+1}) \cdots \int P(x_{n-1}, dx_n) \mathbb{I}\{x \in A\}, \quad 1 \leq k \leq n-1,$$

and $g_P^{(1:1)} := g_P^{(1)}$, and define $g_Q^{(\cdot)}$ similarly using the kernel Q .

Note that $g_P^{(\cdot)}$ and $g_Q^{(\cdot)}$ take values between zero and one and the total variation satisfies $\|\pi - \varpi\| = 2 \sup_{0 \leq f \leq 1} |\pi(f) - \varpi(f)| = 2 \sup_{A \in \mathcal{B}(\mathbf{X})} |\pi(A) - \varpi(A)|$.

$$\begin{aligned} |\mu_X(A) - \mu_Y(A)| &= |\pi(g_P^{(1)}) - \varpi(g_Q^{(1)})| \\ &\leq |\pi(g_Q^{(1)}) - \varpi(g_Q^{(1)})| + |\pi(g_P^{(1)}) - \pi(g_Q^{(1)})| \\ &\leq \frac{1}{2} \|\pi - \varpi\| + |\pi(g_P^{(1)}) - \pi(g_Q^{(1)})|, \end{aligned}$$

showing the claim for $n = 1$. Assume then $n \geq 2$ and observe that we can write $|\pi(g_P^{(1)} - g_Q^{(1)})| = |\mathbb{E}[g_P^{(1)}(X_1) - g_Q^{(1)}(X_1)]|$. We may continue inductively

$$\begin{aligned} &|\mathbb{E}[(g_P^{(1:n-1)} - g_Q^{(1:n-1)})(X_{1:n-1})]| \\ &\leq |\mathbb{E}[(g_P^{(1:n)} - g_Q^{(1:n)})(X_{1:n})]| + \left| \mathbb{E} \left[\int \Delta(X_{n-1}, dx_n) g_Q^{(1:n)}(X_{1:n-1}, x_n) \right] \right|, \end{aligned}$$

where $\Delta(x, dy) := P(x, dy) - Q(x, dy)$, and observe that

$$\begin{aligned} & \left| \mathbb{E} \left[\int \Delta(X_{n-1}, dx_n) g_Q^{(1:n)}(X_{1:n-1}, x_n) \right] \right| \\ & \leq \mathbb{P}(X_{n-1} \notin C) + \sup_{x_{1:n-2} \in \mathcal{X}^{n-2}} \sup_{x_{n-1} \in C} \left| \int \Delta(x_{n-1}, dx_n) g_Q^{(1:n)}(x_{1:n}) \right| \\ & \leq \pi(C^c) + \frac{1}{2} \sup_{x \in C} \|P(x, \cdot) - Q(x, \cdot)\|, \end{aligned}$$

because $|\int \Delta(X_{n-1}, dx_n) g_Q^{(1:n)}(X_{1:n-1}, x_n)| \leq 1$ and $0 \leq g_Q^{(1:n)} \leq 1$. \square

APPENDIX D. LEMMAS FOR SECTION 7

We denote by $n(x) := x/|x|$ the unit vector pointing in the direction of $x \neq 0$ and by $B(x, r) := \{y \in \mathbb{R}^d : |x - y| \leq r\}$ the (closed) Euclidean ball.

Lemma 52. *Assume π satisfies Condition 31, and that $c : \mathbf{X} \rightarrow [1, \infty)$ satisfies $\limsup_{|x| \rightarrow \infty} c(x)e^{-|x|} < \infty$. Then, there exist constants $M, b \in [1, \infty)$ such that*

$$D_x := \left\{ y \in \mathbb{R}^d : \frac{1}{c(x)} \leq \frac{\pi(y)}{\pi(x)} \leq c(x) \right\} \subset B(0, b|x|) \setminus B(0, b^{-1}|x|) \quad \text{for all } |x| \geq M.$$

Proof. Let $c' > \limsup_{|x| \rightarrow \infty} c(x)e^{-|x|}$. Choose any $C \in (4c', \infty)$ and let $M_0 \in [1 \vee \log c', \infty)$ be sufficiently large so that there exists a $\beta_\pi \in (0, 1]$ such that for all $|x| \geq M_0$,

$$c(x) \leq c'e^{|x|}, \quad n(x) \cdot \nabla \log \pi(x) \leq -C \quad \text{and} \quad n(x) \cdot n(\nabla \pi(x)) < -\beta_\pi.$$

Let $\delta \in (0, 1)$, then for any $|x| \geq M_0(1 - \delta)^{-1}$ and all $z = tn(x)$ with $|t| \leq \delta$, we have

$$(55) \quad \left| \log \frac{\pi(x+z)}{\pi(x)} \right| = |t| \int_0^1 |n(x + \lambda z) \cdot \nabla \log \pi(x + \lambda z)| d\lambda \geq C|t|.$$

Now, if $|x| > aM_0$ where $a := \exp(2\pi \tan(\arccos(\beta_\pi)))$, then [28, Lemma 22] implies

$$(56) \quad \{y \in \mathbb{R}^d : \pi(y) = \pi(x)\} \subset B(0, a|x|) \setminus B(0, a^{-1}|x|).$$

Take any $M > 4aM_0$, and choose $|x| \geq M$. Then, the condition (55) implies that any $z = \lambda x \in D_x$, where $\lambda > 0$ satisfies

$$|(\lambda - 1)|x|| \leq C^{-1} \log c(x) \leq C^{-1}(\log(c') + |x|) \leq 2C^{-1}|x|.$$

We deduce that $|\lambda - 1| < 1/2$. Again, using (56), we deduce that the claim holds with $b = 2a$. \square

Lemma 53. *Assume π satisfies Condition 31.*

- (i) *Then, for any constant $\nu \in (0, \infty)$, there exists a constant $b_\nu \in [1, \infty)$ such that for all $x \in \mathbf{X}$, $\{x + z : \frac{\pi(x+z)}{\pi(x)} \geq \nu\} \subset B(0, b_\nu(|x| \vee 1))$.*

Assume also that q satisfies Condition 31.

- (ii) *There exists a constant $\nu \in (0, \infty)$ such that $\inf_{x \in \mathbf{X}} q(\{z : \frac{\pi(x+z)}{\pi(x)} \geq \nu\}) > 0$.*
(iii) *For any constant $\nu \in (0, \infty)$, there exists a constant $M = M(\nu) \in [1, \infty)$ such that $\inf_{|x| \geq M} q(\{z : \frac{\pi(x+z)}{\pi(x)} \geq \nu\}) > 0$.*

Proof. Consider first (i). The existence of such a finite constant follows for x in compact sets by the continuity of π and in the tails by Lemma 52.

The claim (ii) follows on compact sets by the continuity of $\log \pi$, and in the tails as in [13, proof of Theorem 4.3]; the last claim (iii) follows similarly. \square

When the target and the proposal distributions satisfy also Condition 41, we have a decay rate for $q(D_x)$.

Lemma 54. *Assume Condition 41, and assume $\limsup_{|x| \rightarrow \infty} c(x)e^{-|x|} < \infty$. Then, for any $\epsilon' > 0$ there exists a constant $M_0 \in [M, \infty)$ such that for all $|x| \geq M_0$*

$$q(D_x) \leq \epsilon' \frac{\log(c(x))}{|x|^{\rho-1}} \quad \text{where} \quad D_x := \left\{ z \in \mathbb{R}^d : \frac{1}{c(x)} \leq \frac{\pi(x+z)}{\pi(x)} \leq c(x) \right\}.$$

Proof. Lemma 52 implies $b \in [1, \infty)$ and $M' \in [1, \infty)$ such that for all $|x| \geq M'$ the annulus $D_x \subset B(0, b|x|) \setminus B(0, b^{-1}|x|)$. This implies that for any constant $c_\ell \in [1, \infty)$ one can choose $M_\ell \in [M', \infty)$ such that

$$n(x+z) \cdot \nabla \log \pi(x+z) \leq -c_\ell |x+z|^{\rho-1} \quad \text{for all } |x| \geq M_\ell, z \in D_x.$$

Denoting $\ell(x) := \log \pi(x)$, we write

$$D_x = \{z \in \mathbb{R}^d : |\ell(x+z) - \ell(x)| \leq \log c(x)\}.$$

Define the contour surface set $S_{\pi(x)} := \{y \in \mathbb{R}^d : \pi(y) = \pi(x)\}$ and

$$C_{\pi(x)}(\delta) := \{y + tn(y) : y \in S_{\pi(x)}, |t| \leq \delta\}.$$

We will now check that with our conditions, for $|x| \geq M_\ell b$,

$$(57) \quad D_x + x \subset C_{\pi(x)}(\delta_x) \quad \text{where} \quad \delta_x := \frac{b^{\rho-1}}{c_\ell} \cdot \frac{\log c(x)}{|x|^{\rho-1}}.$$

Because $D_x + x = D_y + y$ whenever $\pi(x) = \pi(y)$, it is sufficient to consider $z \in D_x$ such that $z = tn(x)$. As in the proof of Lemma 52,

$$\begin{aligned} |\ell(x+z) - \ell(x)| &= |t| \int_0^1 |n(x+\lambda z) \cdot \nabla \ell(x+\lambda z)| d\lambda \\ &\geq |t| c_\ell |x|^{\rho-1} \int_0^1 \left| 1 + \frac{t}{|x|} \right|^{\rho-1} dt \geq c_\ell b^{-(\rho-1)} |x|^{\rho-1} |t|. \end{aligned}$$

Now $|\ell(x+z) - \ell(x)| \leq \log c(x)$ implies (57).

Write then, by Fubini's theorem,

$$\begin{aligned} q(D_x) &\leq \int_{C_{\pi(x)}(\delta_x) - x} \bar{q}(z) dz \\ &= \int_0^{\bar{q}(0)} \mathcal{L}^d(z \in \mathbb{R}^d : \bar{q}(|z|) \geq t, z \in C_{\pi(x)}(\delta_x) - x) dt \\ &= \int_0^\infty \mathcal{L}^d(z \in \mathbb{R}^d : |z| \leq u, z \in C_{\pi(x)}(\delta_x) - x) |\bar{q}'(u)| du. \end{aligned}$$

Now, [13, proof of Theorem 4.1] shows that for $u \leq |x|/2$,

$$\mathcal{L}^d(C_{\pi(x)}(\delta_x) \cap B(x, u)) \leq \delta_x \left(\frac{|x|+u}{|x|-u} \right)^{d-1} \frac{\mathcal{L}^d(B(x, 3u))}{u} \leq 3^{2d-1} c_d \delta_x u^{d-1},$$

where $c_d = \mathcal{L}^d(B(0, 1))$. By polar integration,

$$\mathcal{L}^d(C_{\pi(x)}(\delta_x)) \leq c_d \sup_{y \in S_{\pi(x)}} \int_{|y|-\delta_x}^{|y|+\delta_x} r^{d-1} dr \leq 2c_d b^{d-1} \delta_x |x|^{d-1} \leq 4c_d b^{d-1} \delta_x u^{d-1},$$

where the latter inequality holds for $u \geq |x|/2$. We obtain

$$q(D_x) \leq c' \delta_x \left(1 + \int_0^\infty u^d |\hat{q}'(u)| du \right),$$

and because \hat{q} is monotone decreasing, integration by substitution yields

$$\int_0^M u^d |\hat{q}'(u)| du = d \int_0^M u^{d-1} \hat{q}(u) du - M^d \hat{q}(M) \leq d c_d^{-1} \int \hat{q}(x) dx < \infty.$$

We deduce $q(D_x) \leq c'' \delta_x$, and conclude by choosing c_ℓ sufficiently large. \square

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